Lecture 24: Bases & Coordinates for abstract vector spaces

Recall $B = \{\vec{v}_1, \dots, \vec{v}_r\}$ is a basis for $\mathbb{V}$ (= abst. vector space) if

(1) B spans $\mathbb{V}$, ie. $\mathbb{V} = Sp(\vec{v}_1, \dots, \vec{v}_r)$

$$= \{\alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r : \alpha_1, \dots, \alpha_r \text{ in } \mathbb{R}\}$$

&

(2) B is linearly independent, ie the only solution to

$$\vec{0} = \alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r \quad \text{is} \quad \alpha_1 = \dots = \alpha_r = 0.$$

Examples: (Lecture 23)

1. $Mat_{2\times3}$ has basis $\{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$ (6 elem)
2. $Mat_{m\times n}$ —— $\{E_{i,j} \quad \begin{matrix} i=1,\dots m \\ j=1,\dots n \end{matrix}\}$ ($m \cdot n$ elem)
3. $\mathcal{P}_d = \{a_0 + a_1 x + \dots + a_d x^d\}$ has basis $\{1, x, x^2, \dots, x^d\}$
   $a_0, \dots, a_d$ in $\mathbb{R}$ (d+1 elem)

Key fact: All basis of a vector space $\mathbb{V}$ have the same number of elements. We define this number as the dimension of $\mathbb{V}$.

⚠ $C[0,1]$ has no basis.

ALGORITHM for computing basis from spanning sets (same as in $\mathbb{R}^n$)

INPUT: $S = \{\vec{v}_1, \ldots, \vec{v}_r\}$ a spanning set for $\mathbb{V}$

OUTPUT: A subset $\mathcal{B}$ of S that is a basis for $\mathbb{V}$.

Step 1: Pick S and ask if S is li/ld.

- If S is LI, output S
- Otherwise, use a nontrivial relation to write one $\vec{v}_i$ as a lin. comb of the remaining vectors.

$$S_{new} = S \setminus \{\vec{v}_i\} = \text{"remove } \vec{v}_i \text{ from } S\text{"}$$

Step 2: Repeat Step 1 with S_{new}, etc.

Example. $S = \{1, (x+1)^2, (x-1)^2, x\}$

Coordinate Systems from bases

Example 1 $\mathbb{R}^n$ has coordinate system induced by $B = \{\vec{e}_1, \dots, \vec{e}_n\}$

Example 2 $\mathrm{Mat}_{m\times n}$ has coord. system induced by $B = \{E_{ij} \;\; \begin{matrix} i=1,\dots m \\ j=1,\dots n \end{matrix}\}$

Example 3 $\mathcal{P}_d$ has coord system induced by the basis of monomials
$B = \{1, x, x^2, \dots, x^d\}$

Summary: If $B = \{\vec{v}_1, \dots, \vec{v}_p\}$ is a basis for $\mathbb{R}^n$ then we can identify $\mathbb{V}$ with $\mathbb{R}^p$, using the "coordinate system" induced by B.

Theorem: Given a vector space $\mathbb{V}$ with basis $B = \{\vec{v}_1, \dots, \vec{v}_p\}$, each vector $\vec{v}$ in $\mathbb{V}$ can be uniquely represented as a linear comb of $\{\vec{v}_1, \dots, \vec{v}_p\}$; meaning $\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v}_p$ and $\alpha_1, \dots, \alpha_p$ are unique.

We call $\boxed{[\vec{v}]_B := \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} \text{ in } \mathbb{R}^p}$ the coordinates of $\vec{v}$ with respect to B

Consequence: We can identify $\mathbb{V}$ with $\mathbb{R}^p$ via $\vec{v} \longleftrightarrow [\vec{v}]_B$

Moreover, we have a map $\Psi : \mathbb{V} \longrightarrow \mathbb{R}^p$ that satisfies

$$\vec{v} \longmapsto [\vec{v}]_B$$

(1) Ψ is a bijection (1-to-1 map, $\Psi^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}\right) = a_1\vec{v}_1 + \cdots + a_p\vec{v}_p$.)

(2) Ψ respects the linear structure on both $\mathbb{V}$ & $\mathbb{R}^p$ (both are vector spaces)

- $\boxed{\Psi(\vec{v}+\vec{w}) = \Psi(\vec{v}) + \Psi(\vec{w})}$

- $\boxed{\Psi(a\vec{v}) = a\Psi(v)}$

Examples

(1) $\left[\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \end{pmatrix}\right]_{\{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}}$

$\left[\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \end{pmatrix}\right]_{\{E_{11}, E_{21}, E_{12}, E_{22}, E_{13}, E_{23}\}}$

(2) $\mathbb{W} = Sp(E_{11}, E_{13}, E_{22}) \longleftrightarrow \mathbb{R}^3$

(3) $\mathcal{P}_2 = \{ a + bx + cx^2 \}$ $\qquad \mathcal{P}_2 \longleftrightarrow \mathbb{R}^3$

$[a + bx + cx^2]_{\{1, x, x^2\}}$

(4) $\mathbb{W} = \mathcal{P}_2$ subspace of $\mathcal{P}_3$ $\quad [a + bx + cx^2]_{\{1, x, x^2, x^3\}}$

Lemma: Identification of coordinates behaves well with respect to addition & scalar multiplication

if $B = \{\vec{v}_1, \ldots, \vec{v}_p\}$

1. $[\vec{0}]_B =$
2. $[\vec{v} + \vec{w}]_B =$
3. $[\alpha \vec{v}]_B =$

Q: What else can we do with coordinates? A Decide li, spanning, bases!

Theorem 2: Fix V a vector space with basis $B = \{\vec{v}_1, \ldots, \vec{v}_p\}$. Consider a set $S = \{\vec{w}_1, \ldots, \vec{w}_r\}$ of vectors in V & write $T = \{[\vec{w}_1]_B, \ldots, [\vec{w}_r]_B\}$ for the vectors in $\mathbb{R}^p$.

Then
1. A vector $\vec{v}$ is in $Sp(\vec{w}_1, \ldots, \vec{w}_r)$ if, and only if, $[\vec{v}]_B$ in $Sp([\vec{w}_1]_B, \ldots, [\vec{w}_r]_B)$
2. The set S is li if, and only if, T is li in $\mathbb{R}^p$.

Consequence 1: All bases for V have the same number of elements.
(because this is true for $\mathbb{R}^p$) Call this number = dimension of V.

Consequence 2: Fix V a vector space of dimension p. Then

1. A set of $p+1$ or more vectors in V is linearly dependent.
2. Any set of $p-1$ or fewer ______________ cannot span V.
3. ______________ p linearly indep vectors in V is a basis for V.
4. ______________ p vectors in V that spans V ______________.

Why? This is true for $\mathbb{R}^p$ (see Lecture 17)

Example: $S = \{ \underset{\vec{v}_1}{1}, \underset{\vec{v}_2}{(x+1)^2}, \underset{\vec{v}_3}{(x-1)^2}, \underset{\vec{v}_4}{x} \}$ in $\mathcal{P}_2$

$\dim \mathcal{P}_2 = 3$

$B = \{ 1, x, x^2 \}$

$$\boxed{-\vec{v}_2 + \vec{v}_3 + 4\vec{v}_4 = \vec{0}}$$

& $\begin{array}{c}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ [\vec{v}_1]_B \quad [\vec{v}_2]_B \quad [\vec{v}_3]_B \quad [\vec{v}_4]_B \end{array} \sim \begin{array}{c}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & -1/4 \end{bmatrix} \\ \text{REF} \end{array}$

- $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is ld & any pair of these vectors is li.

- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ is li in $\mathbb{R}^3$, so it's a basis for $\mathbb{R}^3$.