Lecture 24: Bases & Coordinates for abstract vector spaces

Recall $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_r\}$ is a basis for $\mathbb{V}$ (= abst. vector space) if

(1) $\mathcal{B}$ spans $\mathbb{V}$, ie. $\mathbb{V} = Sp(\vec{v}_1, \dots, \vec{v}_r)$

$$= \{\alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r : \alpha_1, \dots, \alpha_r \text{ in } \mathbb{R}\}$$

&

(2) $\mathcal{B}$ is linearly independent, ie the only solution to

$$\vec{0} = \alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r \quad \text{is} \quad \alpha_1 = \dots = \alpha_r = 0.$$

Examples: (Lecture 23)

1. $Mat_{2\times 3}$ has basis $\{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$ (6 elem)
2. $Mat_{m\times n}$ ——— $\{E_{i,j} \quad i=1,\dots,m;\ j=1,\dots,n\}$ (m·n elem)
3. $\mathcal{P}_d = \{a_0 + a_1 x + \dots + a_d x^d\ ;\ a_0, \dots, a_d \text{ in } \mathbb{R}\}$ has basis $\{1, x, x^2, \dots, x^d\}$ (d+1 elem)

Key fact: All basis of a vector space $\mathbb{V}$ have the same number of elements. We define this number as the dimension of $\mathbb{V}$.

⚠ $C[0,1]$ has no basis.

ALGORITHM for computing basis from spanning sets (same as in $\mathbb{R}^n$)

INPUT: $S = \{\vec{v}_1, \dots, \vec{v}_r\}$ a spanning set for $\mathbb{V}$

OUTPUT: A subset $\mathcal{B}$ of S that is a basis for $\mathbb{V}$.

Step 1: Pick S and ask if S is li/ld.

- If S is LI, output S
- Otherwise, use a nontrivial relation to write one $\vec{v}_i$ as a lin. comb of the remaining vectors.

$$S_{new} = S \setminus \{\vec{v}_i\} = \text{"remove } \vec{v}_i \text{ from } S\text{"}$$

Step 2: Repeat Step 1 with S_{new}, etc.

Example. $S = \{1, (x+1)^2, (x-1)^2, x\}$ spans $\mathcal{P}_2 = Sp(1, x, x^2)$ because 1 in $Sp(S)$, x in $Sp(S)$, $x^2 = (x-1)^2 - 2x - 1$ in $Sp(S)$.

- S is LD. Unique relation: $\boxed{0 = (x+1)^2 - (x-1)^2 - 4x}$

Opt 1: $S_{new} = \{1, (x+1)^2, x\}$, LI. Opt 2 $S_{new} = \{1, (x-1)^2, x\}$, LI. Opt 3 $S_{new} = \{1, (x-1)^2, (x+1)^2\}$ LI

Coordinate Systems from bases

Example ① $\mathbb{R}^n$ has coordinate system induced by $B = \{\vec{e}_1, \dots, \vec{e}_n\}$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{e}_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{e}_2} + \cdots + x_n \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\vec{e}_3}$$

coordinates from B

$$\rightsquigarrow \left[\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right]_B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^6$$

("our usual coordinate system for $\mathbb{R}^n$ comes from the standard basis B")

Example ② $\mathrm{Mat}_{m\times n}$ has coord. system induced by $B = \{E_{ij} \ \ \begin{smallmatrix} i=1,\dots,m \\ j=1,\dots,n \end{smallmatrix}\}$

$$A = (a_{ij}) = a_{11}E_{11} + \cdots + a_{1n}E_{1n} + a_{21}E_{21} + \cdots + a_{mn}E_{mn}.$$

Ex $m=2$, $n=3$

$$\left[\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right]_B = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} \text{ in } \mathbb{R}^6$$

$$\mathrm{Mat}_{2\times 3} \longleftrightarrow \mathbb{R}^6$$

$$E_{ij} \longleftrightarrow e_k$$

$$k = i(n-1)+j$$

Example 3 $\mathcal{P}_d$ has coord system induced by the basis of monomials $B = \{1, x, x^2, \dots, x^d\}$

$$[a_0 + a_1 x + a_2 x^2 + \dots + a_d x^d]_B = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{bmatrix} \text{ in } \mathbb{R}^{d+1}$$

$$\mathcal{P}_d \longleftrightarrow \mathbb{R}^{d+1}$$
$$x^i \longmapsto e_{i+1}$$

Summary: If $B = \{\vec{v}_1, \dots, \vec{v}_p\}$ is a basis for $\mathbb{R}^n$ then we can identify $\mathbb{V}$ with $\mathbb{R}^p$, using the "coordinate system" induced by B.

Theorem: Given a vector space $\mathbb{V}$ with basis $B = \{\vec{v}_1, \dots, \vec{v}_p\}$, each vector $\vec{v}$ in $\mathbb{V}$ can be uniquely represented as a linear comb of $\{\vec{v}_1, \dots, \vec{v}_p\}$; meaning $\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v}_p$ and $\alpha_1, \dots, \alpha_p$ are unique.

We call $\boxed{[\vec{v}]_B := \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} \text{ in } \mathbb{R}^p}$ the coordinates of $\vec{v}$ with respect to B

Why? • $\alpha_1 \dots \alpha_p$ exist because B spans

• Uniqueness: If 2 solutions $\alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v}_p = \vec{v} = \beta_1 \vec{v}_1 + \dots + \beta_p \vec{v}_p$.

Then $(\alpha_1 - \beta_1)\vec{v}_1 + \dots + (\alpha_p - \beta_p)\vec{v}_p$ & B li forces $\alpha_1 = \beta_1, \dots, \alpha_p = \beta_p$.

Consequence: We can identify $\mathbb{V}$ with $\mathbb{R}^p$ via $\vec{v} \longleftrightarrow [\vec{v}]_B$

Moreover, we have a map $\Psi : \mathbb{V} \longrightarrow \mathbb{R}^p$ that satisfies
$$\vec{v} \longmapsto [\vec{v}]_B$$

(1) Ψ is a bijection (1-to-1 map, $\Psi^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}\right) = a_1\vec{v}_1 + \cdots + a_p\vec{v}_p$.)

(2) Ψ respects the linear structure on both $\mathbb{V}$ & $\mathbb{R}^p$ (both are vector spaces)

- $\boxed{\Psi(\vec{v}+\vec{w}) = \Psi(\vec{v}) + \Psi(\vec{w})}$

$$\begin{aligned} & \vec{v} = \alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p \\ + \quad & \vec{w} = \beta_1 \vec{v}_1 + \cdots + \beta_p \vec{v}_p \\ \hline & \vec{v}+\vec{w} = (\alpha_1+\beta_1)\vec{v}_1 + \cdots + (\alpha_p+\beta_p)\vec{v}_p \end{aligned}$$

& $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_p \end{bmatrix} = \begin{bmatrix} \alpha_1+\beta_1 \\ \vdots \\ \alpha_p+\beta_p \end{bmatrix}$

- $\boxed{\Psi(a\vec{v}) = a\Psi(v)}$

$a\vec{v} = a(\alpha_1\vec{v}_1 + \cdots + \alpha_p\vec{v}_p) = (a\alpha_1)\vec{v}_1 + \cdots + (a\alpha_p)\vec{v}_p$ & $a\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} a\alpha_1 \\ \vdots \\ a\alpha_p \end{bmatrix}$

Examples

(1) $$\left[\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \end{pmatrix}\right]_{\{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}} = \begin{bmatrix} a \\ 0 \\ b \\ 0 \\ c \\ 0 \end{bmatrix}$$

$$\left[\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \end{pmatrix}\right]_{\{E_{11}, E_{21}, E_{12}, E_{22}, E_{13}, E_{23}\}} = \begin{bmatrix} a \\ 0 \\ 0 \\ c \\ b \\ 0 \end{bmatrix}$$

So order of elements in the basis matters!

(2) $$W = Sp(E_{11}, E_{13}, E_{22}) \longleftrightarrow \mathbb{R}^3$$

$$A = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [A]_{\{E_{11}, E_{13}, E_{22}\}}$$

$$= aE_{11} + bE_{13} + cE_{22}$$

(3) $$\mathcal{P}_2 = \{a + bx + cx^2\} \qquad \mathcal{P}_2 \longleftrightarrow \mathbb{R}^3$$

$$[a + bx + cx^2]_{\{1, x, x^2\}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

(4) $W = \mathcal{P}_2$ subspace of $\mathcal{P}_3$ $\quad [a + bx + cx^2]_{\{1, x, x^2, x^3\}} = \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix}$

$$W \longleftrightarrow \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ in } \mathbb{R}^4 : x_4 = 0 \right\}$$

Lemma: Identification of coordinates behaves well with respect to addition & scalar multiplication

(1) $[\vec{0}]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^p$. if $B = \{\vec{v}_1, \ldots, \vec{v}_p\}$

(2) $[\vec{v} + \vec{w}]_B = [\vec{v}]_B + [\vec{w}]_B$ in $\mathbb{R}^p$

(3) $[\alpha \vec{v}]_B = \alpha [\vec{v}]_B$

Q: What else can we do with coordinates? A Decide li, spanning, bases!

Theorem 2: Fix $\mathbb{V}$ a vector space with basis $B = \{\vec{v}_1, \ldots, \vec{v}_p\}$ Consider a set $S = \{\vec{w}_1, \ldots, \vec{w}_r\}$ of vectors in $\mathbb{V}$. & write $T = \{[\vec{w}_1]_B, \ldots, [\vec{w}_r]_B\}$ for the vectors in $\mathbb{R}^p$.

Then (1) A vector $\vec{v}$ is in $Sp(\vec{w}_1, \ldots, \vec{w}_r)$ if, and only if, $[\vec{v}]_B$ in $Sp([\vec{w}_1]_B, \ldots, [\vec{w}_r]_B)$

(2) The set S is li if, and only if, T is li in $\mathbb{R}^p$.

Why? (1) $\vec{v} = \alpha_1 \vec{w}_1 + \cdots + \alpha_r \vec{w}_r$ & take $[\;]_B$. By Lemma we get

$[\vec{v}]_B = \alpha_1 [\vec{w}_1]_B + \cdots + \alpha_r [\vec{w}_r]_B$. We can reverse the process!

(2) Take $\vec{v} = \vec{0}$ & use (1).

Consequence 1: All bases for $\mathbb{V}$ have the same number of elements.
(because this is true for $\mathbb{R}^p$) Call this number = dimension of $\mathbb{V}$.

Why? Take B & B' two bases for $\mathbb{V}$ with $p = |B| \leq |B'| = p'$

Use $[\ ]_{B'}$ & $S = B = \{\vec{v}_1, \ldots, \vec{v}_p\}$ spans $\mathbb{V}$, so

$\{[\vec{v}_1]_{B'}, \ldots, [\vec{v}_p]_{B'}\}$ spans $\mathbb{R}^{p'}$ Then: $p \geq p'$ & so $\boxed{p = p'}$
(p vectors)

(because $p' \geq p$ & $p \geq p'$)

Consequence 2: Fix $\mathbb{V}$ a vector space of dimension p. Then

① A set of $p+1$ or more vectors in $\mathbb{V}$ is linearly dependent.

② Any set of $p-1$ or fewer ____________ cannot span $\mathbb{V}$.

③ ____________ p linearly indep vectors in $\mathbb{V}$ is a basis for $\mathbb{V}$.

④ ____________ p vectors in $\mathbb{V}$ that spans $\mathbb{V}$ ____________.

Why? This is true for $\mathbb{R}^p$ (see Lecture 17)

<u>Example</u>: $S = \{ \underset{\vec{v}_1}{1}, \underset{\vec{v}_2}{(x+1)^2}, \underset{\vec{v}_3}{(x-1)^2}, \underset{\vec{v}_4}{x} \}$ in $\mathcal{P}_2$ $\qquad \dim \mathcal{P}_2 = 3$

$B = \{1, x, x^2\}$

$[1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[(x+1)^2]_B = [x^2+2x+1]_B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$[(x-1)^2]_B = [x^2-2x+1]_B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $[x]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

- S is l.d because $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is l.d in $\mathbb{R}^3$.
- Find relations in S using relations for T in $\mathbb{R}^3$:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_2]{} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{bmatrix} \xrightarrow[R_3 \to \frac{R_3}{-4}]{} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1/4 \end{bmatrix}$$

$$\xrightarrow[\substack{R_2 \to R_2 - R_3 \\ R_1 \to R_1 - R_3}]{} \begin{bmatrix} 1 & 1 & 0 & 1/4 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & -1/4 \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_2]{} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & -1/4 \end{bmatrix}}_{\text{REF}}$$

x_1, x_2, x_3 dep
x_4 indep

$$\begin{cases} x_1 = 0 \\ x_2 = -\frac{1}{4} x_4 \\ x_3 = \frac{1}{4} x_4 \end{cases}$$

All relations come from $-[\vec{v}_2]_B + [\vec{v}_3]_B + 4[\vec{v}_4]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad (x_4 = 4)$

So $\boxed{-\vec{v}_2 + \vec{v}_3 + 4\vec{v}_4 = \vec{0}}$ "generates all relations in S"!

$$\boxed{-\vec{v}_2 + \vec{v}_3 + 4\vec{v}_4 = \vec{0}}$$

&

$$\begin{array}{cccc} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ [\vec{v}_1]_B \quad [\vec{v}_2]_B \quad [\vec{v}_3]_B \quad [\vec{v}_4]_B \end{array} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & -1/4 \end{bmatrix}$$

REF

- $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is ld & any pair of these vectors is li.

Same is true for $\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$

- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ is li in $\mathbb{R}^3$, so it's a basis for $\mathbb{R}^3$. Therefore $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $V = \mathcal{P}_2$.

In particular x^2 in $\mathrm{Sp}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

$$\Big\updownarrow [\ ]_B$$

$$\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \text{ in } \mathrm{Sp}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 1 & 1 & 2 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 1 & 1 & 2 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = (-2)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \rightsquigarrow \boxed{x^2 = (-2)\,1 + 1\,(x+1)^2 + 1\,(x-1)^2}$$

same scalars!