Lecture 25 : §5.7 Linear Transformations for abstract vector spaces

Main idea: We can easily generalize lin. transf $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ to

$$T : \mathbb{V} \longrightarrow \mathbb{W}$$

when $\dim \mathbb{V} = n$ & $\dim \mathbb{W} = m$ because we have $+$ & $\cdot$ in both $\mathbb{V}$ & $\mathbb{W}$.

Definition: Fix $\mathbb{V}$ & $\mathbb{W}$ two abstract vector spaces & $T : \mathbb{V} \longrightarrow \mathbb{W}$ a map (assignment, $T(\vec{v}) = \vec{w}$ in $\mathbb{W}$.) $\vec{v} \longmapsto \vec{w}$

We say T is a linear transformation if

(1) $T(\underbrace{\vec{v} + \vec{u}}_{\text{sum in } \mathbb{V}}) = \underbrace{T(\vec{v}) + T(\vec{u})}_{\text{sum in } \mathbb{W}}$ for all $\vec{v}, \vec{u}$ in $\mathbb{V}$

(2) $T(\underbrace{\alpha \vec{v}}_{\text{scalar mult in } \mathbb{V}}) = \underbrace{\alpha T(\vec{v})}_{\text{scalar mult in } \mathbb{W}}$ for all $\vec{v}$ in $\mathbb{V}$. α scalar.

Examples

$T: V \to W$ lin. transf

1. $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, usual linear transf from § 3.7.

2. $T: \mathcal{P}_2 \longrightarrow \mathbb{R}$ $\quad T(P(x)) = P(1)$ is a linear transf

Q: Explicit formula for T?

$$T(a + bx + cx^2) = a + b + c$$

3. $T: C_{[0,1]} \longrightarrow \mathbb{R}$ is linear (same idea).

$$f \longmapsto f(1)$$

④ Taking coordinates with respect to a fixed basis $\mathcal{B}$ for $\mathbb{V}$ ($\dim \mathbb{V} = p$)

$$T : \mathbb{V} \longrightarrow \mathbb{R}^p \qquad \text{is linear (Lecture 24)}$$
$$\vec{v} \longmapsto [\vec{v}\,]_{\mathcal{B}}$$

- More examples in Lecture Notes.
- Combine ② & ④ via compositions

Key fact: We'll do the same for $\mathbb{V} \xrightarrow{T} \mathbb{W}$ $\qquad \dim \mathbb{V} = n$, $\dim \mathbb{W} = m$

After choosing bases $\mathcal{B}_{\mathbb{V}}$ & $\mathcal{B}_{\mathbb{W}}$: get $\mathbb{R}^n \xrightarrow{\tilde{T}} \mathbb{R}^n$ $\qquad$ linear

$\boxed{\text{Basic Properties}}$

Theorem 1: Fix a basis $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ for V & any list of n many vectors $\vec{w}_1, \dots, \vec{w}_n$ in W. Then we can find a <u>unique</u> linear transformation $T: V \longrightarrow W$ with

$$\begin{cases} T(\vec{v}_1) = \vec{w}_1 \\ \quad\vdots \\ T(\vec{v}_n) = \vec{w}_n \end{cases}$$

Why? Same idea as with $\mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Write $\vec{v}$ in V as $\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n$ $\alpha_1, \dots, \alpha_n$ are fixed

- By construction:
  - linear
  - $T(\vec{v}_i) = \vec{w}_i$ because $[v_i]_B = \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array}\begin{array}{l} \\ \\ \leftarrow \text{ith spot} \\ \\ \\ \\ \end{array} = e_i$

    (so $\alpha_i = 1$ & rest $\alpha_j = 0$)

Example! Find a linear transformation $T: \mathcal{P}_3 \longrightarrow \mathcal{P}_2$ with

$T(1) = 2+x$, $T(x) = x - x^2$, $T(x^2) = 5 - 10x$ & $T(x^3) = 2$

So $\tilde{T} : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ linear

$$[P]_{B_V} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$V = \mathcal{P}_3 \rightsquigarrow B_V = \{1, x, x^2, x^3\}$

$W = \mathcal{P}_2 \rightsquigarrow B_W = \{1, x, x^2\}$

A = matrix representing T with respect to the bases B_V & B_W

gives $[T(v)]_{B_W} = A\,[v]_{B_V}$ (more, next time!)

Null Space & Range

$T: \mathbb{V} \to \mathbb{W}$ lin transf

Use same definitions as for $\tilde{T}: \mathbb{R}^n \longrightarrow \mathbb{R}^m$. linear transf

Def 1: The Null Space of T is $\mathcal{N}(T) = \{\vec{v} \text{ in } \mathbb{V} : T(\vec{v}) = \vec{0} \text{ in } \mathbb{W}\}$

Def 2 The Range of T: is $R(T) = \{\vec{w} \text{ in } \mathbb{W} : \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \text{ in } \mathbb{V}\}$

(= image of the map T)

Theorem 1: (1) $\mathcal{N}(T)$ is a subspace of $\mathbb{V}$

(2) $R(T)$ ________________ $\mathbb{W}$

Theorem 2: Fix $T: V \longrightarrow W$ linear transf, $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ basis for V.

Then: (1) $R(T) = Sp\left(T(\vec{v}_1), \dots, T(\vec{v}_n)\right)$ so $\dim R(T) \leq n$

(2) $N(T) = \{\vec{0}_W\}$ if and only if $\{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$ li

(equiv, $\dim R(T) = n$)

Example: $T: \mathcal{P}_2 \longrightarrow \mathbb{R}^2$ $\quad T(p) = \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix}$

Proposition 1 Fix $T: V \longrightarrow W$ linear transf

(1) $T(\vec{u}) = T(\vec{v})$ if and only if $(\vec{u} - \vec{v})$ is in $\mathcal{N}(T)$

(2) T is injective (meaning $T(\vec{u}) = T(\vec{v})$ forces $\vec{u} = \vec{v}$) if, and only if, $\mathcal{N}(T) = \{\vec{0}_V\}$.

Def: $\text{nullity}(T) = \dim \mathcal{N}(T)$

$\text{rank}(T) = \dim \mathcal{R}(T)$

Rank-Nullity Thm: If $T: V \longrightarrow W$ is linear & $\dim V = n$ then $\boxed{n = \text{nullity}(T) + \text{rank}(T)}$.

Proof: See the Lecture Notes (optional reading but very insightful)

Consequence: A $m \times n$ matrix, then $\text{rank}(A) + \text{nullity}(A) = \#\text{cols } A$

Why? A represents $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ & $\text{rank}(A) = \text{rank}(T)$
$\text{nullity}(A) = \text{nullity}(T)$.

Examples

① $T : \mathcal{P}_2 \longrightarrow \mathbb{R}^2$ $\quad T(p) = \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix}$ linear $\quad R(T) = \mathbb{R}^2$

dim 3 $\qquad$ dim 2 $\qquad \mathcal{N}(T) = Sp\left((1-x)^2\right)$

② $T : \mathrm{Mat}_{2\times 3} \longrightarrow \mathcal{P}_4$ $\quad T\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) = (a_{12}+a_{23})\,x^4 + (2a_{22}+3a_{13})x^3 + (a_{11}-a_{23}).$

dim 6 $\qquad$ 5

- T is linear $\Longleftrightarrow$ $\tilde{T} : \mathbb{R}^6 \longrightarrow \mathbb{R}^5$

$B_V = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$

$B_W = \{1, x, x^2, x^3, x^4\}$

- $\mathcal{N}(T) = ?$ $\qquad$ Need $(a_{12}+a_{23})\,x^4 + (2a_{22}+3a_{13})\,x^3 + (a_{11}-a_{23})\,1 = \vec{0}$ in $\mathcal{P}_4$

$T : \mathrm{Mat}_{2\times 3} \longrightarrow \mathcal{P}_4 \quad T\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) = (a_{12}+a_{23})x^4 + (2a_{22}+3a_{13})x^3 + (a_{11}-a_{23}).$

dim 6 5

$\text{nullity}(T) = 3$