Lecture 25 : §5.7 Linear Transformations for abstract vector spaces

Main idea: We can easily generalize lin. transf $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ to

$$T: \mathbb{V} \longrightarrow \mathbb{W}$$

where $\dim \mathbb{V} = n$ & $\dim \mathbb{W} = m$ because we have $+$ & $\cdot$ in both $\mathbb{V}$ & $\mathbb{W}$.

Definition: Fix $\mathbb{V}$ & $\mathbb{W}$ two abstract vector spaces & $T: \mathbb{V} \longrightarrow \mathbb{W}$, $\vec{v} \longmapsto \vec{w}$
a map (assignment, $T(\vec{v}) = \vec{w}$ in $\mathbb{W}$.)

We say T is a linear transformation if

(1) $T(\underbrace{\vec{v}+\vec{u}}_{\text{sum in } \mathbb{V}}) = T(\vec{v}) \underbrace{+}_{\text{sum in } \mathbb{W}} T(\vec{u})$ for all $\vec{v}, \vec{u}$ in $\mathbb{V}$

(2) $T(\underbrace{\alpha \vec{v}}_{\text{scalar mult in } \mathbb{V}}) = \underbrace{\alpha\, T(\vec{v})}_{\text{scalar mult in } \mathbb{W}}$ for all $\vec{v}$ in $\mathbb{V}$, α scalar.

$\boxed{\text{Examples}}$

$T: V \longrightarrow W$ lin. transf

1. $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, usual linear transf from §3.7.

2. $T: \mathcal{P}_2 \longrightarrow \mathbb{R}$ $\quad T(P_{(x)}) = P_{(1)}$ is a linear Transf

- $(P+Q)_{(x)} = P_{(x)} + Q_{(x)}$ so $(P+Q)_{(1)} = P_{(1)} + Q_{(1)}$ ✓
  ↳ def of $+$ in $\mathcal{P}_2$
- $(\alpha P)_{(x)} = \alpha P(x)$ so $(\alpha P)_{(1)} = \alpha P_{(1)}$. ✓
  ↳ def of $\cdot$ in $\mathcal{P}_2$

Q: Explicit formula for T?

$$T(a+bx+cx^2) = a+b+c \quad \rightsquigarrow \quad \tilde{T}: \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$[a+bx+cx^2]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \longmapsto a+b+c$$

$B = \{1, x, x^2\}$ linear!

3. $T: C_{[0,1]} \longrightarrow \mathbb{R}$ is linear (same idea).
$\quad f \longmapsto f_{(1)}$

④ Taking coordinates with respect to a fixed basis B for $\mathbb{V}$ $(\dim \mathbb{V} = p)$

$$T: \mathbb{V} \longrightarrow \mathbb{R}^p$$
$$\vec{v} \longmapsto [\vec{v}]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}$$

is linear (Lecture 24)

if $\vec{v} = \alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p$

$B = \{\vec{v}_1, \ldots, \vec{v}_p\}$

- More examples in Lecture Notes.

- Combine ② & ④ via compositions

$$P \longmapsto P(1) \quad (= a+b+c \text{ if } P(x) = a+bx+cx^2)$$

$$\begin{tikzcd}
P \arrow[d, mapsto] & \mathcal{P}_2 \arrow[r, "T"] \arrow[d, "F"'] & \mathbb{R} \arrow[d, equal] \\
{[P]_{\{1,x,x^3\}}} & \mathbb{R}^3 \arrow[r, "\tilde{T}"'] & \mathbb{R}
\end{tikzcd}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longmapsto a+b+c$$

<u>Conclude</u>: $T(P) = \tilde{T} \circ F(P) = \tilde{T}(F(P))$

for every polynomial P in $\mathcal{P}_2$

<u>Key fact</u>: We'll do the same for

After choosing bases $B_{\mathbb{V}}$ & $B_{\mathbb{W}}$: get

$$\begin{tikzcd}
\mathbb{V} \arrow[r, "T"] \arrow[d] & \mathbb{W} \arrow[d] \\
\mathbb{R}^n \arrow[r, "\tilde{T}"] & \mathbb{R}^n
\end{tikzcd}$$

$\vec{v} \mapsto [\vec{v}]_{B_{\mathbb{V}}}$

$\vec{w} \mapsto [\vec{w}]_{B_{\mathbb{W}}}$

$\dim \mathbb{V} = n$

$\dim \mathbb{W} = m$

linear

Basic Properties

Theorem 1: Fix a basis $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ for V & any list of n many vectors $\vec{w}_1, \dots, \vec{w}_n$ in W. Then we can find a <u>unique</u> linear transformation $T: V \longrightarrow W$ with
$$\begin{cases} T(\vec{v}_1) = \vec{w}_1 \\ \vdots \\ T(\vec{v}_n) = \vec{w}_n \end{cases}.$$

Why? Same idea as with $\mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Write $\vec{v}$ in V as $\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n$ $\quad$ $\alpha_1, \dots, \alpha_n$ are fixed

so $T(\vec{v}) \underset{T \text{ linear}}{=} \alpha_1 T(\vec{v}_1) + \dots + \alpha_n T(\vec{v}_n)$

$= \alpha_1 \vec{w}_1 + \dots + \alpha_n \vec{w}_n$ $\quad \rightsquigarrow$ unique way to define T.

- By construction:
  - linear
  - $T(\vec{v}_i) = \vec{w}_i$ because $[v_i]_B = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e_i$ (ith spot)

    (so $\alpha_i = 1$ & rest $\alpha_j = 0$)

Example! Find a linear transformation $T : \mathbb{P}_3 \longrightarrow \mathbb{P}_2$ with

$T(1) = 2+x$, $T(x) = x - x^2$, $T(x^2) = 5-10x$ & $T(x^3) = 2$

Solution $T(a + bx + cx^2 + dx^3) = aT(1) + bT(x) + cT(x^2) + dT(x^3)$

$= a(2+x) + b(x-x^2) + c(5-10x) + d\,2$

$= (2a + 5c + 2d) + (a + b - 10c)x + (-b)x^2$

So $\tilde{T} : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ linear $\leadsto$ we get a 3x4 Matrix

$$[P]_{B_V} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longmapsto \begin{bmatrix} 2a+5c+2d \\ a+b-10c \\ -b \end{bmatrix} = [T(P)]_{B_W} \qquad A = \begin{bmatrix} 2 & 0 & 5 & 2 \\ 1 & 1 & -10 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$V = \mathbb{P}_3 \leadsto B_V = \{1, x, x^2, x^3\}$

$W = \mathbb{P}_2 \leadsto B_W = \{1, x, x^2\}$

A = matrix representing T with respect to the bases B_V & B_W.

gives $[T(v)]_{B_W} = A\,[v]_{B_V}$ (more, next Time!)

Null Space & Range

$T : \mathbb{V} \to \mathbb{W}$ lin transf

Use same definitions as for $\tilde{T} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear transf

Def 1: The <u>Null Space of T</u> is $\mathcal{N}(T) = \{ \vec{v} \text{ in } \mathbb{V} : T(\vec{v}) = \vec{0} \text{ in } \mathbb{W} \}$

Def 2 The <u>Range of T</u>: is $\mathcal{R}(T) = \{ \vec{w} \text{ in } \mathbb{W} : \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \text{ in } \mathbb{V} \}$

(= image of the map T)

Theorem 1: (1) $\mathcal{N}(T)$ is a subspace of $\mathbb{V}$

(2) $\mathcal{R}(T)$ ________________ $\mathbb{W}$

Lecture 22

Why? <u>Key</u>: $T(\vec{0}_{\mathbb{V}}) = \vec{0}_{\mathbb{W}}$. $\left(T(\vec{0}_{\mathbb{V}}) = T(0 \cdot \vec{0}_{\mathbb{V}}) = 0 \, \underbrace{T(\vec{0}_{\mathbb{V}})}_{\text{in } \mathbb{W}} = \vec{0}_{\mathbb{W}} \right)$

So (S1) holds for both $\mathcal{N}(T)$ & $\mathcal{R}(T)$.

(S2) follows from T respects +

(S3) ________________________ .

Theorem 2: Fix $T : \mathbb{V} \longrightarrow \mathbb{W}$ linear transf, $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ basis for $\mathbb{V}$.

Then: (1) $R(T) = Sp(T(\vec{v}_1), \ldots, T(\vec{v}_n))$ so $\dim R(T) \leq n$

(2) $\mathcal{N}(T) = \{\vec{0}_{\mathbb{W}}\}$ if and only if $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ li

(equiv, $\dim R(T) = n$)

Why? (1) $\vec{v} = \alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n$

$$T(\vec{v}) = \alpha_1 T(\vec{v}_1) + \cdots + \alpha_n T(\vec{v}_n) \text{ so in } Sp(T(\vec{v}_1), \ldots, T(\vec{v}_n))$$

Since all $T(\vec{v}_i)$ are in $R(T)$, (1) follows.

(2) Only one soln for $\vec{0}_{\mathbb{W}} = \alpha_1 T(\vec{v}_1) + \cdots + \alpha_n T(\vec{v}_n)$ is li

This translates to only one option for $T(\vec{v}) = \vec{0}_{\mathbb{W}}$, namely $\vec{v} = \vec{0}_{\mathbb{V}}$

Example: $T : \mathcal{P}_2 \longrightarrow \mathbb{R}^2$ $\quad T(p) = \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix}$ is linear

- $\mathcal{N}(T) = \left\{ a + bx + cx^2 : \begin{cases} a+b+c = 0 \\ b+2c = 0 \end{cases} \right\} = \{c(1-2x+x^2)\}_{c \text{ in } \mathbb{R}} = Sp(1-2x+x^2)$
- $R(T) = \left\{ \begin{bmatrix} a+b+c \\ b+2c \end{bmatrix} : a, b, c \right\} = \left\{ a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} = Sp\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \mathbb{R}^2$

Proposition 1 Fix $T : \mathbb{V} \longrightarrow \mathbb{W}$ linear transf

(1) $T(\vec{u}) = T(\vec{v})$ if and only if $(\vec{u} - \vec{v})$ is in $\mathcal{N}(T)$

(2) T is injective (meaning $T(\vec{u}) = T(\vec{v})$ forces $\vec{u} = \vec{v}$) if, and only if, $\mathcal{N}(T) = \{\vec{0}_{\mathbb{V}}\}$.

Def: $\text{nullity}(T) = \dim \mathcal{N}(T)$

$\text{rank}(T) = \dim \mathcal{R}(T)$

Rank-Nullity Thm: If $T : \mathbb{V} \longrightarrow \mathbb{W}$ is linear & $\dim \mathbb{V} = n$ then

$$\boxed{n = \text{nullity}(T) + \text{rank}(T)}.$$

Proof: See the Lecture Notes (optional reading but very insightful)

Consequence: A $m \times n$ matrix, then $\text{rank}(A) + \text{nullity}(A) = \# \text{cols } A$

Why? A represents $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ & $\text{rank}(A) = \text{rank}(T)$

$\text{nullity}(A) = \text{nullity}(T)$.

$\boxed{\text{Examples}}$

① $T : \mathcal{P}_2 \longrightarrow \mathbb{R}^2$ $\quad T(p) = \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix}$ linear $\quad R(T) = \mathbb{R}^2$

dim 3 $\qquad$ dim 2 $\qquad \mathcal{N}(T) = Sp\{(1-x)^2\}$

$$3 = 1 + 2 = \dim \mathcal{N}(T) + \dim R(T).$$

② $T : \underset{\text{dim } 6}{Mat_{2\times 3}} \longrightarrow \underset{5}{\mathcal{P}_4}$ $\quad T\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) = (a_{12}+a_{23})x^4 + (2a_{22}+3a_{13})x^3 + (a_{11}-a_{23}).$

• T is linear $\rightsquigarrow$ $\tilde{T} : \mathbb{R}^6 \longrightarrow \mathbb{R}^5$ $\quad \tilde{T}\left(\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}\right) = \begin{bmatrix} a_{11}-a_{23} \\ 0 \\ 0 \\ 2a_{22}+3a_{13} \\ a_{12}+a_{23} \end{bmatrix} = \begin{bmatrix} 1&0&0&0&0&-1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&3&0&2&0 \\ 1&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} a_{11} \\ \vdots \\ a_{23} \end{bmatrix}$

$B_V = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$

$B_W = \{1, x, x^2, x^3, x^4\}$

• $\mathcal{N}(T) = ?$ $\quad$ Need $(a_{12}+a_{23})\underline{x^4} + (2a_{22}+3a_{13})\underline{x^3} + (a_{11}-a_{23})\underline{1} = \vec{0}$ in $\mathcal{P}_4$

3 li polynomials (part of a basis!)

So $\begin{cases} a_{12}+a_{23} = 0 \\ 2a_{22}+3a_{13} = 0 \\ a_{11}-a_{23} = 0 \end{cases}$ $\quad \begin{array}{c} \begin{bmatrix} 0&1&0&0&0&-1 \\ 0&0&3&0&2&0 \\ 1&0&0&0&0&1 \end{bmatrix} \\ a_{11}\ a_{12}\ a_{13}\ a_{21}\ a_{22}\ a_{23} \end{array} \overset{G\text{-}J}{\sim} \begin{bmatrix} 1&0&0&0&0&-1 \\ 0&1&0&0&0&1 \\ 0&0&1&0&2/3&0 \end{bmatrix}$ REF

a_{11}, a_{12}, a_{13} dep. vars

a_{21}, a_{22}, a_{23} indep vars

So $\quad A = \begin{bmatrix} a_{23} & -a_{23} & -\frac{2}{3}a_{22} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = a_{23}\underbrace{\begin{bmatrix} 1&-1&0 \\ 0&0&1 \end{bmatrix}}_{A_1} + a_{22}\underbrace{\begin{bmatrix} 0&0&-2/3 \\ 0&1&0 \end{bmatrix}}_{A_2} + a_{21}\underbrace{\begin{bmatrix} 0&0&0 \\ 1&0&0 \end{bmatrix}}_{A_3}$ $\quad \mathcal{N}(T) = Sp(A_1, A_2, A_3)$

nullity$(T) = 3$

$T : \underset{\dim\ 6}{\mathrm{Mat}_{2\times 3}} \longrightarrow \underset{5}{\mathcal{P}_4}$ $\quad T\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) = (a_{12}+a_{23})x^4 + (2a_{22}+3a_{13})x^3 + (a_{11}-a_{23}).$

$\mathrm{nullity}(T) = 3$ so $\mathrm{rank}(T) = 6-3 = 3.$

$R(T) = \mathrm{Sp}\left(T(E_{11}), \ldots, T(E_{23}) \right)$

$T\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = 1$

$T\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = x^4$

$T\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = 3x^3$

$T\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) = 0$

$T\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = 2x^3$

$T\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = x^4-1$

$\rightsquigarrow \underset{\dim\ 3}{R(T)} = \mathrm{Sp}(1, x^4, 3x^3, 0, 2x^3, x^4-1)$

$= \mathrm{Sp}(1, x^4, x^3)$

$[1]_{B_W} = e_1$

$[x^4]_{B_W} = e_5$

$[x^3]_{B_W} = e_4$

$B_W = \{1, x, x^2, x^3, x^4\}$

Note: $\tilde{T} : \mathbb{R}^6 \longrightarrow \mathbb{R}^5$ $\quad \tilde{T}\left(\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} \\ \vdots \\ a_{23} \end{bmatrix}$

$R(\tilde{T}) = \mathrm{Sp}(e_1 - e_5, 3e_4, 2e_4, -e_1 + e_5) = \mathrm{Sp}(e_1, e_5, e_4)$