Lecture 26: §5.8 Operations with linear transformations

Last Time: • Defined linear transformations $T: V \longrightarrow W$ V, W v.sp.
• Defined $\mathcal{N}(T)$, $\mathcal{R}(T)$ & saw Rank-Nullity Thm:
$$\dim V = \dim \mathcal{N}(T) + \dim \mathcal{R}(T).$$

TODAY'S GOAL: Describe operations between linear transformations

• Correspondence to keep in mind:

$$\boxed{T: \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ linear} \longleftrightarrow A \text{ of size } m \times n.}$$

• **Summary**:

(1) $\mathrm{Mat}_{m\times n}$, $\{T: \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ linear}\}$, $\{T: V \longrightarrow W \text{ linear}\}$ are ALL vector spaces (we have addition & scalar multiplication satisfying the 10 nice properties from Lecture 21)

(2) We have composition / multiplication for some pairs of linear transf / matrices.

Operations	Matrices	$T: \mathbb{R}^n \to \mathbb{R}^m$ linear	$T: \mathbb{V} \to \mathbb{W}$ linear
(I) ADDITION	$A + C$ matrix $(A+C)_{ij} = A_{ij} + C_{ij}$	$F+G: \mathbb{R}^n \to \mathbb{R}^m$ linear $(F+G)(\vec{v}) =$	$F+G: \mathbb{V} \to \mathbb{W}$ linear $(F+G)(\vec{v}) =$
(II) SCALAR MULTIPLICATION	$\alpha \cdot A$ matrix $(\alpha A)_{ij} = \alpha A_{ij}$	$\alpha T: \mathbb{R}^n \to \mathbb{R}^m$ linear $(\alpha T)(\vec{v}) =$	$\alpha T: \mathbb{V} \to \mathbb{W}$ linear $(\alpha T)(\vec{v}) =$
(III) MULTIPLICATION vs. COMPOSITION	A in $Mat_{m \times n}$ C in $Mat_{s \times m}$ Then CA in $Mat_{s \times n}$ $(CA)_{ij} = C_{i1}A_{1j} + \dots + C_{im}A_{mj}$	$F: \mathbb{R}^n \to \mathbb{R}^m$ linear $G: \mathbb{R}^m \to \mathbb{R}^s$ linear Then $G \circ F: \mathbb{R}^n \to \mathbb{R}^s$ linear $G \circ F(\vec{v}) =$	$F: \mathbb{V} \to \mathbb{W}$ linear $G: \mathbb{W} \to \mathbb{U}$ linear Then $G \circ F: \mathbb{V} \to \mathbb{U}$ linear $(G \circ F)(\vec{v}) =$

Examples

① $T_1 : \overset{\text{dim } 3}{\mathcal{P}_2} \longrightarrow \mathbb{R}$, $T_2 : \mathcal{P}_2 \longrightarrow \mathbb{R}$ both are linear transf (last time)

$p \longmapsto p(1)$, $p \longrightarrow p'(2)$

New function: $T_1 + T_2 : \mathcal{P}_2 \longrightarrow \mathbb{R}$ is also linear.
(Addition)

$T_1: \mathcal{P}_2 \longrightarrow \mathbb{R}$, $T_2: \mathcal{P}_2 \longrightarrow \mathbb{R}$

$P \longmapsto P(1)$ $\quad$ $P \longrightarrow P'(2)$

$T_1(a+bx+cx^2) = a+b+c$

$T_2(a+bx+cx^2) = b+4c$

New function

(Scalar mult)

$3T_1 : \mathcal{P}_2 \longrightarrow \mathbb{R}$

Observation:

($T: \mathbb{V} \to \mathbb{W}$ linear, α scalar)

$\mathcal{N}(T) = \mathcal{N}(\alpha T)$ whenever $\alpha \neq 0$

$\mathcal{R}(T) = \mathcal{R}(\alpha T)$ ——————

If $\alpha = 0$ $\quad$ $0 \cdot T$ = constant zero function $\mathbb{V} \longrightarrow \mathbb{W}$, $\vec{v} \longmapsto \vec{0}_{\mathbb{W}}$ so $\mathcal{N}(0 \cdot T) = \mathbb{V}$, $\mathcal{R}(0 \cdot T) = \{\vec{0}_{\mathbb{W}}\}$

Observation 2: In general, we should not expect any relation between

- $\mathcal{N}(T_1)$, $\mathcal{N}(T_2)$ & $\mathcal{N}(T_1 + T_2)$
- $\mathcal{R}(T_1)$, $\mathcal{R}(T_2)$ & $\mathcal{R}(T_1 + T_2)$

② $T_1 : Mat_{2\times 3} \longrightarrow P_3$ linear , $T_2 : P_3 \longrightarrow \mathbb{R}^2$ linear

$A \longmapsto a_{11}x^3 + (a_{12}-a_{13})x^2 + a_{23}$ $\qquad a+bx+cx^2+dx^3 \longrightarrow \begin{bmatrix} d-c \\ d-b+a \end{bmatrix}$

New function (Composition) $T_2 \circ T_1 :$

$$\begin{tikzcd} Mat_{2\times 3} \arrow[r, "T_1"] \arrow[rr, bend right, "T_2 \circ T_1"'] & P_3 \arrow[r, "T_2"] & \mathbb{R}^2 \end{tikzcd}$$

Q1: Are $N(T_1)$ & $N(T_2 \circ T_1)$ related? Both are subspaces of $Mat_{2\times 3}$.

A

Observation 3: This is true in general!

If $T_1 : V \longrightarrow W$ & $T_2 : W \longrightarrow U$ are linear, then any $\vec{v}$ in $N(T_1)$ is automatically in $N(T_2 \circ T_1)$. In symbols $N(T_1) \subseteq N(T_2 \circ T_1)$

$T_1 : \mathrm{Mat}_{2\times 3} \longrightarrow \mathcal{P}_3$ linear , $T_2 : \mathcal{P}_3 \longrightarrow \mathbb{R}^2$ linear

$A \longmapsto a_{11}x^3 + (a_{12}-a_{13})x^2 + a_{23}$ $\qquad a+bx+cx^2+dx^3 \longrightarrow \begin{bmatrix} d-c \\ d-b+a \end{bmatrix}$

New function (Composition)

$$\begin{tikzcd} T_2 \circ T_1 : \mathrm{Mat}_{2\times 3} \arrow[r, "T_1"] \arrow[rr, bend right, "T_2\circ T_1"'] & \mathcal{P}_3 \arrow[r, "T_2"] & \mathbb{R}^2 \end{tikzcd}$$

$$A \longmapsto \underset{\underbrace{a_{11}}_{d}x^3 + \underbrace{(a_{12}-a_{13})}_{c}x^2 + \underbrace{a_{23}}_{a}}{P(x)} \longrightarrow \begin{bmatrix} a_{11}-(a_{12}-a_{13}) \\ a_{11}-0+a_{23} \end{bmatrix}$$

$(b=0)$ $\qquad$ linear in (a_{ij})

Q1: Are $R(T_2)$ & $R(T_2 \circ T_1)$ related? Both are subspaces of $\mathbb{R}^2$

A

Observation 3: This is true in general!

If $T_1 : V \longrightarrow W$ & $T_2 : W \longrightarrow U$ are linear, then any $\vec{u}$ in $R(T_2 \circ T_1)$ is automatically in $R(T_2)$. In symbols $R(T_2 \circ T_1) \subseteq R(T_2)$

$T_1 : Mat_{2\times 3} \longrightarrow \mathcal{P}_3$ linear , $T_2 : \mathcal{P}_3 \longrightarrow \mathbb{R}^2$ linear

$A \longmapsto a_{11}x^3 + (a_{12}-a_{13})x^2 + a_{23}$ $\quad a+bx+cx^2+dx^3 \longrightarrow \begin{bmatrix} d-c \\ d-b+a \end{bmatrix}$

$T = T_2 \circ T_1 : Mat_{2\times 3} \longrightarrow \mathbb{R}^3$ $\quad T(A) = \begin{bmatrix} a_{11}-a_{12}+a_{13} \\ a_{11}+a_{23} \end{bmatrix}$

. Let's check our claims

(1) $\mathcal{N}(T_1) \subseteq \mathcal{N}(T_2 \circ T_1)$

(2) $\mathcal{R}(T_2 \circ T_1) \subseteq \mathcal{R}(T_2)$

Surjective / Onto Linear Transformations

(use dim to check!)

Def: A linear transf $T: V \longrightarrow W$ is onto or surjective if $R(T) = W$

Example ① $T_2: P_3 \longrightarrow \mathbb{R}^2$ $\quad T_2(a+bx+cx^2+dx^3) = \begin{bmatrix} d-c \\ d-b+a \end{bmatrix}$ is onto

since $R(T_2) = \mathbb{R}^2$. (last slide)

Example ② $T: P_3 \longrightarrow P_2$ $\quad T(p(x)) = p'(x)$

Example ③ : $T: P_3 \longrightarrow P_3$ $\quad T(p(x)) = p'(x)$

Invertible Transformations

Def: A linear transf $T: V \longrightarrow W$ is <u>invertible</u> if we can find $S: W \longrightarrow V$ linear transf with (1) $T \circ S: W \longrightarrow W = id_W$, $\vec{w} \longmapsto \vec{w}$

(2) $S \circ T: V \longrightarrow V = id_V$, $\vec{v} \longmapsto \vec{v}$

Alt name: Isomorphism.

Prop: If T is invertible, S is unique. Call it T^{-1}.

Special case $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\quad T(\vec{x}) = A\vec{x}$

T is invertible if and only if $m = n$ & A is invertible

($S: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with matrix A^{-1} $\quad (S(\vec{y}) = A^{-1}\vec{y})$.)

Main example: Fix $\mathbb{V}$ with basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_p\}$

$$T: \mathbb{V} \longrightarrow \mathbb{R}^p \quad \text{is invertible with} \quad T^{-1}: \mathbb{R}^p \longrightarrow \mathbb{V}$$

$$\vec{v} \longmapsto [\vec{v}]_{\mathcal{B}} \qquad\qquad \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} \longmapsto$$

Both linear & (1) $T \circ T^{-1} = id_{\mathbb{R}^p} : \mathbb{R}^p \longrightarrow \mathbb{R}^p$ (2) $T^{-1} \circ T = id_{\mathbb{V}} : \mathbb{V} \longrightarrow \mathbb{V}$

Non-example 1 $T: \mathbb{P}_2 \longrightarrow \mathbb{R}$ linear but not invertible

$$p \longmapsto p(1)$$

Non-example 2 $T: \mathbb{R} \longrightarrow \mathbb{R}^2$ $x = \begin{bmatrix} 2x \\ x \end{bmatrix}$ linear but not invertible

Main Theorem: $T: \mathbb{V} \longrightarrow \mathbb{W}$ linear transf is invertible if and only if $\mathcal{N}(T) = \{0_{\mathbb{V}}\}$ AND $R(T) = \mathbb{W}$

Observe: By the Rank-Nullity: $\dim \mathbb{V} = \underset{\substack{\| \\ 0}}{\text{nullity}}(T) + \text{rank}\, T = \dim \mathbb{W}$

Why?

- Define $S: \mathbb{W} \longrightarrow \mathbb{V}$

$$\vec{w} \longmapsto \vec{v}$$