Lecture 26: §5.8 Operations with linear transformations

Last Time: • Defined linear transformations $T: V \longrightarrow W$ $\quad V, W$ v.sp.
• Defined $N(T)$, $R(T)$ & saw Rank-Nullity Thm:
$$\dim V = \dim N(T) + \dim R(T).$$

TODAY'S GOAL: Describe operations between linear transformations

• Correspondence to keep in mind:

$$\boxed{T: \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ linear} \longleftrightarrow A \text{ of size } m \times n.}$$

• **Summary**:

(1) $\mathrm{Mat}_{m\times n}$, $\{T: \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ linear}\}$, $\{T: V \longrightarrow W \text{ linear}\}$
are ALL vector spaces (we have addition & scalar multiplication satisfying the 10 nice properties from Lecture 21)

(2) We have composition / multiplication for some pairs of linear transf / matrices.

Operations	Matrices	$T:\mathbb{R}^n \to \mathbb{R}^m$ linear	$T:\mathbb{V} \to \mathbb{W}$ linear
(I) ADDITION	$A + C$ matrix $(A+C)_{ij} = A_{ij} + C_{ij}$	$F+G : \mathbb{R}^n \to \mathbb{R}^m$ linear $(F+G)(\vec{v}) = F(\vec{v}) + G(\vec{v})$	$F+G : \mathbb{V} \to \mathbb{W}$ linear $(F+G)(\vec{v}) = F(\vec{v}) \underset{\text{in } \mathbb{W}}{+} G(\vec{v})$
(II) SCALAR MULTIPLICATION	$\alpha \cdot A$ matrix $(\alpha A)_{ij} = \alpha A_{ij}$	$\alpha T : \mathbb{R}^n \to \mathbb{R}^m$ linear $(\alpha T)(\vec{v}) = \alpha \cdot T(\vec{v})$	$\alpha T : \mathbb{V} \to \mathbb{W}$ linear $(\alpha T)(\vec{v}) = \alpha \underset{\text{in } \mathbb{W}}{\cdot} T(\vec{v})$
(III) MULTIPLICATION vs. COMPOSITION	A in $\mathrm{Mat}_{m\times n}$ C in $\mathrm{Mat}_{s\times m}$ Then CA in $\mathrm{Mat}_{s\times n}$ $(CA)_{ij} = C_{i1}A_{1j} + \cdots + C_{im}A_{mj}$	$F : \mathbb{R}^n \to \mathbb{R}^m$ linear $G : \mathbb{R}^m \to \mathbb{R}^s$ linear Then $G \circ F : \mathbb{R}^n \to \mathbb{R}^s$ linear $G\circ F(\vec{v}) = G(\underbrace{F(\vec{v})}_{\text{in } \mathbb{R}^m})$	$F : \mathbb{V} \to \mathbb{W}$ linear $G : \mathbb{W} \to \mathbb{U}$ linear Then $G \circ F : \mathbb{V} \to \mathbb{U}$ linear $(G\circ F)(\vec{v}) = G(\underbrace{F(\vec{v})}_{\text{in } \mathbb{W}})$

If $[F]=A$, $[G]=C$ then $[G\circ F] = CA$

Examples

① $T_1 : \overset{\text{dim } 3}{\mathcal{P}_2} \longrightarrow \mathbb{R}$, $T_2 : \mathcal{P}_2 \longrightarrow \mathbb{R}$ both are linear transf (last time)

$P \longmapsto P(1)$ $\qquad P \longrightarrow P'(2)$

Why? $\cdot\ T_1(a+bx+cx^2) = a+b+c$

$\cdot\ T_2(a+bx+cx^2) = (b+2cx)\big|_{x=2} = b+4c$

$\Big\}$ are linear in $[P]_{\{1,x,x^2\}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

New function: (Addition) $T_1+T_2 : \mathcal{P}_2 \longrightarrow \mathbb{R}$ is also linear.

$P(x) \longmapsto P(1)+P'(2)$

$(T_1+T_2)(a+bx+cx^2) = (a+b+c) + b+4c = a+2b+5c$

$\cdot\ \mathcal{N}(T_1+T_2) = \{ a+bx+cx^2 : a+2b+5c = 0 \}$ $\qquad a = -2b-5c$

$= \{(-2b-5c)+bx+cx^2\} = \{ b(-2+x)+c(-5+x^2)\}$

$= Sp(-2+x, -5+x^2)$ $\qquad$ dim $= 2$ $\rightsquigarrow \mathcal{R}(T_1+T_2) = \mathbb{R}$

$\cdot\ \mathcal{N}(T_1) = \{ a+bx+cx^2 : a+b+c = 0\} = \{(-b-c)+bx+cx^2\}$

$= \{ b(-1+x)+c(-1+x^2)\} = Sp(-1+x, -1+x^2)$ $\qquad$ dim 2

$\cdot\ \mathcal{N}(T_2) = \{ a+bx+cx^2 : b+4c = 0\} = Sp(1, (-4x+x^2))$ $\qquad$ dim 2

No relation between 3 null-spaces. $\qquad \mathcal{R}(T_1) = \mathcal{R}(T_2) = \mathbb{R}$

$T_1: \mathcal{P}_2 \longrightarrow \mathbb{R}$, $T_2: \mathcal{P}_2 \longrightarrow \mathbb{R}$

$P \longmapsto P(1)$ $\qquad$ $P \longrightarrow P'(2)$

$T_1(a+bx+cx^2) = a+b+c$

$T_2(a+bx+cx^2) = b+4c$

New function (Scalar mult)

$3T_1 : \mathcal{P}_2 \longrightarrow \mathbb{R}$ is linear $(3T_1)_{(a+bx+cx^2)} = 3a+3b+3c$

$P \longmapsto 3P(1)$

$\mathcal{N}(3T_1) = \mathcal{N}(T_1)$ & $\mathcal{R}(3T_1) = \mathcal{R}(T_1)$

$[3T(p) = 3(T(p))]$ $\qquad$ [because $3T_1(p) = T_1(3P)$]

Observation: $\mathcal{N}(T) = \mathcal{N}(\alpha T)$ whenever $\alpha \neq 0$

($T: \mathbb{V} \longrightarrow \mathbb{W}$ linear, α scalar) $\mathcal{R}(T) = \mathcal{R}(\alpha T)$ ————

If $\alpha = 0$ $\quad 0 \cdot T$ = constant zero function $\mathbb{V} \longrightarrow \mathbb{W}$, $\vec{v} \longmapsto \vec{0}_{\mathbb{W}}$ so $\mathcal{N}(0 \cdot T) = \mathbb{V}$, $\mathcal{R}(0 \cdot T) = \{\vec{0}_{\mathbb{W}}\}$

Observation 2: In general, we should not expect any relation between

- $\mathcal{N}(T_1)$, $\mathcal{N}(T_2)$ & $\mathcal{N}(T_1+T_2)$
- $\mathcal{R}(T_1)$, $\mathcal{R}(T_2)$ & $\mathcal{R}(T_1+T_2)$

② $T_1 : Mat_{2\times 3} \longrightarrow \mathcal{P}_3$ linear , $T_2 : \mathcal{P}_3 \longrightarrow \mathbb{R}^2$ linear

$A \longmapsto a_{11}x^3 + (a_{12}-a_{13})x^2 + a_{23}$, $a+bx+cx^2+dx^3 \longmapsto \begin{bmatrix} d-c \\ d-b+a \end{bmatrix}$

New function (Composition) $T_2 \circ T_1 : Mat_{2\times 3} \xrightarrow{T_1} \mathcal{P}_3 \xrightarrow{T_2} \mathbb{R}^2$

$$\begin{tikzcd} Mat_{2\times 3} \arrow[r, "T_1"] \arrow[rr, bend right, "T_2 \circ T_1"'] & \mathcal{P}_3 \arrow[r, "T_2"] & \mathbb{R}^2 \end{tikzcd}$$

$$A \longmapsto \underset{\underbrace{a_{11}}_{d}x^3 + \underbrace{(a_{12}-a_{13})}_{c}x^2 + \underbrace{a_{23}}_{a}}{P(x)} \longmapsto \begin{bmatrix} a_{11}-(a_{12}-a_{13}) \\ a_{11} - 0 + a_{23} \end{bmatrix}$$ linear in (a_{ij})

$(b=0)$

Q1: Are $\mathcal{N}(T_1)$ & $\mathcal{N}(T_2 \circ T_1)$ related? Both are subspaces of $Mat_{2\times 3}$.

A YES! If $\vec{v}$ in $\mathcal{N}(T_1)$, then $T_1(\vec{v}) = \mathbb{0}$ in $\mathcal{P}_3$

Applying T_2 gives $\underbrace{T_2(T_1(\vec{v}))}_{= T_2 \circ T_1(\vec{v})} = T_2(\mathbb{0}) = \mathbb{0}$ in $\mathbb{R}^2$

so $\vec{v}$ is in $\mathcal{N}(T_2 \circ T_1)$

Observation 3: This is true in general!

If $T_1 : V \longrightarrow W$ & $T_2 : W \longrightarrow U$ are linear, then any $\vec{v}$ in $\mathcal{N}(T_1)$ is automatically in $\mathcal{N}(T_2 \circ T_1)$. In symbols $\mathcal{N}(T_1) \subseteq \mathcal{N}(T_2 \circ T_1)$

$T_1 : \mathrm{Mat}_{2\times 3} \longrightarrow \mathcal{P}_3$ linear , $T_2 : \mathcal{P}_3 \longrightarrow \mathbb{R}^2$ linear

$A \longmapsto a_{11}x^3 + (a_{12} - a_{13})x^2 + a_{23}$, $a + bx + cx^2 + dx^3 \longrightarrow \begin{bmatrix} d-c \\ d-b+a \end{bmatrix}$

<u>New function</u> $T_2 \circ T_1 : \mathrm{Mat}_{2\times 3} \xrightarrow{T_1} \mathcal{P}_3 \xrightarrow{T_2} \mathbb{R}^2$

(Composition)

$$\begin{tikzcd} \mathrm{Mat}_{2\times 3} \arrow[r, "T_1"] \arrow[rr, bend right, "T_2 \circ T_1"'] & \mathcal{P}_3 \arrow[r, "T_2"] & \mathbb{R}^2 \end{tikzcd}$$

$$A \longmapsto P(x) \longrightarrow \begin{bmatrix} a_{11} - (a_{12} - a_{13}) \\ a_{11} - 0 + a_{23} \end{bmatrix}$$

$$P(x) = \underbrace{a_{11}}_{d} x^3 + \underbrace{(a_{12} - a_{13})}_{c} x^2 + \underbrace{a_{23}}_{a} \qquad (b=0)$$

linear in (a_{ij})

Q1: Are $R(T_2)$ & $R(T_2 \circ T_1)$ related? Both are subspaces of $\mathbb{R}^2$

A YES! If $\vec{u}$ is in $R(T_2 \circ T_1)$, write $\vec{u} = T_2 \circ T_1(A)$ for some $A_{2\times 3}$

Then $\vec{u} = T_2(\underbrace{T_1(A)}_{P \text{ in } \mathcal{P}_3})$ so $\vec{u} = T_2(P)$ is in $R(T_2)$

<u>Observation 3</u>: This is true in general!

If $T_1 : V \longrightarrow W$ & $T_2 : W \longrightarrow U$ are linear, then any $\vec{u}$ in $R(T_2 \circ T_1)$ is automatically in $R(T_2)$ <u>In symbols</u> $R(T_2 \circ T_1) \subseteq R(T_2)$

$T_1 : Mat_{2\times 3} \longrightarrow \mathcal{P}_3$ linear , $T_2 : \mathcal{P}_3 \longrightarrow \mathbb{R}^2$ linear

$A \longmapsto a_{11}x^3 + (a_{12}-a_{13})x^2 + a_{23}$ $\qquad a+bx+cx^2+dx^3 \longrightarrow \begin{bmatrix} d-c \\ d-b+a \end{bmatrix}$

$T = T_2 \circ T_1 : Mat_{2\times 3} \longrightarrow \mathbb{R}^3 \qquad T(A) = \begin{bmatrix} a_{11}-a_{12}+a_{13} \\ a_{11}+a_{23} \end{bmatrix}$

. Let's check our claims

(1) $\mathcal{N}(T_1) \subseteq \mathcal{N}(T_2 \circ T_1)$ ✓

(2) $\mathcal{R}(T_2 \circ T_1) \subseteq \mathcal{R}(T_2)$

(1) . $\mathcal{N}(T_1) = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} : \begin{cases} a_{11} = 0 \\ a_{12}-a_{13} = 0 \\ a_{23} = 0 \end{cases} \right\} = \left\{ \begin{bmatrix} 0 & a_{13} & a_{13} \\ a_{21} & a_{22} & 0 \end{bmatrix} \right\} = Sp(E_{12}+E_{13}, E_{21}, E_{22})$

$\mathcal{N}(T_2 \circ T_1) = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} : \begin{cases} a_{11}-a_{12}+a_{13} = 0 \\ a_{11}+a_{23} = 0 \end{cases} \right\} = \left\{ \begin{bmatrix} a_{11} & a_{12} & -a_{11}+a_{12} \\ a_{21} & a_{22} & -a_{11} \end{bmatrix} \right\}$

↳ dep vars

$= Sp(E_{11} - E_{13} - E_{23}, \underbrace{E_{12}+E_{13}, E_{21}, E_{22}}_{\text{they span } \mathcal{N}(T_1)})$

(2) $\mathcal{R}(T_2 \circ T_1) = Sp(T_2 \circ T_1(E_{11}), T_2 \circ T_1(E_{12}), \ldots, T_2 \circ T_1(E_{23}) \}$

$= Sp \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$

$\mathcal{R}(T_2) = Sp(T_2(1), T_2(x), T_2(x^2), T_2(x^3)) = Sp\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \mathbb{R}^2$

Surjective / Onto Linear Transformations

(use dim to check!)

Def: A linear transf $T: V \longrightarrow W$ is onto or surjective if $R(T) = W$

Example 1 $T_2 : \mathcal{P}_3 \longrightarrow \mathbb{R}^2$ $\quad T_2(a+bx+cx^2+dx^3) = \begin{bmatrix} d-c \\ d-b+a \end{bmatrix}$ is onto since $R(T_2) = \mathbb{R}^2$. (last slide)

Example 2 $T : \mathcal{P}_3 \longrightarrow \mathcal{P}_2$ $\quad T(P_{(x)}) = P'_{(x)}$ is onto

Why? given $Q = a+bx+cx^2$ in $\mathcal{P}_2$ write $P = \int_0^x Q_{(t)}\,dt$.

- Then $P'_{(x)} = Q$ by fundamental Thm of Calculus.
- also $P_{(x)} = ax + \frac{b}{2}x + \frac{c}{3}x^3$ is in $\mathcal{P}_3$ (degree goes up by 1).

Alternative check: $R(T) = Sp\big(T(1), T(x), T(x^2), T(x^3)\big)$

$= Sp(0, 1, 2x, 3x^2)$

$= Sp(1, x, x^2) = \mathcal{P}_2$.

Example 3 : $T : \mathcal{P}_3 \longrightarrow \mathcal{P}_3$ $\quad T(P(x)) = P'(x)$ is NOT onto $(R(T) = \mathcal{P}_2 \neq \mathcal{P}_3)$

Invertible Transformations

Def: A linear transf $T: V \longrightarrow W$ is invertible if we can find $S: W \longrightarrow V$ linear transf with (1) $T \circ S: W \longrightarrow W = id_W$, $\vec{w} \longmapsto \vec{w}$

(2) $S \circ T: V \longrightarrow V = id_V$, $\vec{v} \longmapsto \vec{v}$

Alt name: Isomorphism.

Prop: If T is invertible, S is unique. Call it T^{-1}.

Why? If we have 2 such functions S_1 & S_2 so $\begin{cases} T \circ S_2 = id_W \\ S_1 \circ T = id_V \end{cases}$

$$S_1 = S_1 \circ id_W = S_1 \circ (T \circ S_2) = (S_1 \circ T) \circ (S_2) = id_V \circ S_2 = S_2$$

We conclude $S_1 = S_2 : W \longrightarrow V$ (same function!)

Special case $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $T(\vec{x}) = A\vec{x}$

T is invertible if and only if $m = n$ & A is invertible

($S: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with matrix A^{-1} $(S(\vec{y}) = A^{-1}\vec{y})$)

Main example: Fix V with basis $B = \{\vec{v}_1, \dots, \vec{v}_p\}$

$T: V \longrightarrow \mathbb{R}^p$ is invertible with $T^{-1}: \mathbb{R}^p \longrightarrow V$

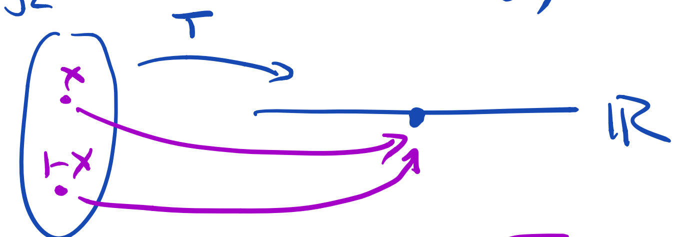
$\vec{v} \longmapsto [\vec{v}]_B$ $\qquad\qquad$ $\begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_p\end{bmatrix} \longmapsto \alpha_1\vec{v}_1 + \dots + \alpha_p\vec{v}_p$

Both linear & (1) $T \circ T^{-1} = id_{\mathbb{R}^p}: \mathbb{R}^p \longrightarrow \mathbb{R}^p$ $\quad$ (2) $T^{-1} \circ T = id_V : V \longrightarrow V$

Non-example (1) $T: \mathbb{P}_2 \longrightarrow \mathbb{R}$ $\quad$ linear but not invertible

$\mathbb{P}_2$ $\qquad p \longmapsto p(1)$

Issue:



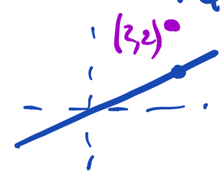
$T(0) = T(1-x) = 0$

Q: How to decide $S(0) = 0$ or $1-x$.

$N(T) \neq \{0_{\mathbb{P}_2}\}$ so T is not injective!

Non-example (2) $T: \mathbb{R} \longrightarrow \mathbb{R}^2$ $\quad$ $x = \begin{bmatrix}2x\\ x\end{bmatrix}$ $\quad$ linear but not invertible

Issue. $R(T) =$ line through $(0,0)$ with dir $\begin{bmatrix}2\\1\end{bmatrix}$ —



Q: What would $\underset{\vec{v}=}{T^{-1}}\left(\begin{bmatrix}2\\2\end{bmatrix}\right)$ be? Then $T(\vec{v}) = \begin{bmatrix}2\\2\end{bmatrix}$ but $(2,2)$ not in the line!

$R(T) \neq \mathbb{R}^2$, so T is not surjective!

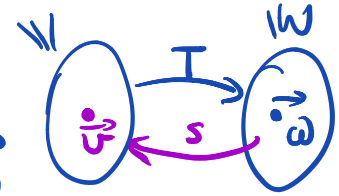
Main Theorem: $T: V \to W$ linear transf is invertible

if and only if $\mathcal{N}(T) = \{0_V\}$ AND $R(T) = W$

Observe: By the Rank-Nullity: $\dim V = \underset{\overset{\shortparallel}{0}}{\text{nullity}}(T) + \text{rank}\,T = \dim W$

Why? • Define $S: W \longrightarrow V$

$\vec{w} \longmapsto \vec{v}$ where $T(\vec{v}) = \vec{w}$



. T surjective, so we have such $\vec{v}$.

. T injective, so there's a unique choice for $\vec{v}$.

• S is linear because

① if $\vec{w}_1 = T(\vec{v}_1)$, $\vec{w}_2 = T(\vec{v}_2)$ then $\vec{w}_1 + \vec{w}_2 = T\underbrace{(\vec{v}_1 + \vec{v}_2)}$ so $S(\vec{w}_1 + \vec{w}_2) = \vec{v}_1 + \vec{v}_2$
$= S(\vec{w}_1) + S(\vec{w}_2)$

② if $\vec{w} = T(\vec{v})$, α scalar, then $\alpha\vec{w} = T(\alpha\vec{v})$ so $S(\alpha\vec{w}) = \alpha\vec{v}$
$= \alpha S(\vec{w})$

• Easy check: $T \circ S = \mathrm{id}_W$ & $S \circ T = \mathrm{id}_V$.