Lecture 27: §5.9 Matrix representations for linear transformations

Recall A linear transf $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is always given by a matrix $A_{m\times n}$

$$A = [\,T(\vec{e}_1), \ldots, T(\vec{e}_n)\,]$$

Ex: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 - x_2 \\ 2x_1 + 4x_2 \\ x_2 \end{bmatrix}$$

TODAY's GOAL: Find a similar way to interpret $T: V \longrightarrow W$ linear transf

Summary:
- If $\dim V = n$ $\quad B_V = \{\vec{v}_1, \ldots, \vec{v}_n\}$ basis for V
- If $\dim W = m$ $\quad B_W = \{\vec{w}_1, \ldots, \vec{w}_m\}$ basis for W

Then

$$\begin{tikzcd}
\vec{v} \arrow[d, mapsto] & V \arrow[r, "T"] \arrow[d, "\cong \text{ iso}"] & W & \vec{w} = \beta_1 \vec{w}_1 + \cdots + \beta_m \vec{w}_m \\
{[\vec{v}]_{B_V}} & \mathbb{R}^n \arrow[r, "\tilde{T}", "\text{matrix } A"'] & \mathbb{R}^m \arrow[u, "\cong"] & \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} \arrow[u, mapsto]
\end{tikzcd}$$

$$\boxed{A = \left[\, [T(\vec{v}_1)]_{B_W} \cdots [T(\vec{v}_n)]_{B_W} \,\right]}$$

$$\rightsquigarrow \boxed{[T(\vec{v})]_{B_W} = A\,[\vec{v}]_{B_V}} \text{ in } \mathbb{R}^m$$

NAME: A = matrix of T relative to the bases B_V & B_W.

$\boxed{\text{Three Fundamental Facts}}$

$T: V \longrightarrow W$ linear

FACT 1 T is completely determined by its values on a basis for V.

FACT 2 If $\dim V = n$ & $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for V, then

$$T: V \longrightarrow \mathbb{R}^n \quad \text{is linear and invertible with} \quad T^{-1}: \mathbb{R}^n \longrightarrow V$$

$$\vec{v} \longmapsto [\vec{v}]_B \qquad\qquad \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \longmapsto \alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n$$

Write $\boxed{T = T_B}$ to emphasize the crucial role of the basis B.

Upshot: By choosing a basis B for V we can always think of V as $\mathbb{R}^n$.

Conclusion 1: If we choose coordinates for V & W by picking a basis for each space, then T can be thought of as $\tilde{T}: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear

FACT 3 Composition of linear maps is linear

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[rr, bend right, "G \circ F"'] & W \arrow[r, "G"] & U \end{tikzcd}$$

$G \circ F(\vec{v}) = G(F(\vec{v}))$

Examples

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[d, "T_{B_W}"] \\ \mathbb{R}^n \arrow[u, "T_{B_V}^{-1}"] \arrow[r, "\tilde{T}"'] & \mathbb{R}^m \end{tikzcd}$$

1 $T : \mathcal{P}_3 \longrightarrow \mathcal{P}_2$

$\quad\quad P_{(x)} \longmapsto P'_{(x)}$

$$T(a+bx+cx^2+dx^3) = b + 2cx + 3dx^2$$

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$$T: \mathrm{Mat}_{2\times 3} \longrightarrow \mathbb{R}^3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A \longmapsto \begin{bmatrix} 2a_{11} - 4a_{22} \\ a_{13} + a_{21} \\ a_{12} - a_{23} \end{bmatrix}$$

$$T: \mathrm{Mat}_{2\times 3} \longrightarrow \mathbb{R}^3 \qquad B_{\mathrm{Mat}_{2\times3}} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A \longmapsto \begin{bmatrix} 2a_{11} - 4a_{22} \\ a_{13} + a_{21} \\ a_{12} - a_{23} \end{bmatrix}$$

Q: What if we chose a different basis for $\mathbb{R}^3$?

$$B'_{\mathbb{R}_3} = \left\{ \overset{\vec{w}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \overset{\vec{w}_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \overset{\vec{w}_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \right\} \qquad \text{New } A = \begin{bmatrix} 2 & 0 & -1 & -1 & -4 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\tilde{T} := T_{B_{\mathrm{Mat}_{2\times3}} B'_{\mathbb{R}^3}} : \mathbb{R}^6 \xrightarrow{(T_{B_{\mathrm{Mat}}})^{-1}} \mathrm{Mat}_{2\times3} \xrightarrow{T} \mathbb{R}^3 \xrightarrow{T_{B'_{\mathbb{R}^3}}} \mathbb{R}^3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix} \longrightarrow \begin{bmatrix} 2x_1 - 4x_5 \\ x_3 + x_4 \\ x_2 - x_6 \end{bmatrix} \longrightarrow \left[\begin{bmatrix} 2x_1 - 4x_5 \\ x_3 + x_4 \\ x_2 - x_6 \end{bmatrix}\right]_{B'_{\mathbb{R}^3}}$$

$$T\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} =$$

$$T\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} =$$

$$T\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} =$$

$$T\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix} =$$

$$T\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} =$$

Matrix Representation

Representation Theorem: Fix $T: V \longrightarrow W$ linear transf with $\dim V = n$ & $\dim W = m$. Pick $B_V = \{\vec{v}_1, \dots, \vec{v}_n\}$ basis for V

$B_W = \{\vec{w}_1, \dots, \vec{w}_m\}$ — W.

Then, T, B_V & B_W gives rise to a linear transformation

$$T_{B_V B_W} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

defined by

$$T_{B_V B_W}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = T_{B_W} \circ T \circ T_{B_V}^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right)$$

$$= \left[T(a_1\vec{v}_1 + \dots + a_n\vec{v}_n)\right]_{B_W}$$

Furthermore, the associated $m \times n$ matrix is obtained as follows:

$$\boxed{[T]_{B_V B_W}} = \Big[\, [T(\vec{v}_1)]_{B_W}, \dots, [T(\vec{v}_n)]_{B_W} \,\Big]$$

Consequence: $T(\vec{v}) = \beta_1 \vec{w}_1 + \dots + \beta_m \vec{w}_m$ where $\underset{m\times 1}{\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}} = \underset{m\times n}{[T]_{B_V B_W}} \underset{n\times 1}{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}$

$\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n$

$\boxed{\text{Examples}}$

1. $T : \mathcal{P}_3 \longrightarrow \mathcal{P}_2$

$\quad P_{(x)} \longmapsto P'_{(x)}$

$B_{\mathcal{P}_3} = \{1, x, x^2, x^3\}$

$B_{\mathcal{P}_2} = \{1, x, x^2\}$

Q $[T]_{B_{\mathcal{P}_3} B_{\mathcal{P}_2}} = ?$

(2) $T: \mathcal{P}_2 \longrightarrow \mathbb{R}^2$

$a+bx+cx^2 \mapsto \begin{bmatrix} c-b \\ 2c+a \end{bmatrix}$

$B_{\mathcal{P}_2} = \{1, x, x^2\}$

$B_{\mathbb{R}^2} = \{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{w}_1}, \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{w}_2} \}$

$T(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$T(x^2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- $[T(3+4x+5x^2)]_{B_{\mathbb{R}^2}}$

So $T(3+4x+5x^2) =$

(3) $T: \mathrm{Mat}_{2\times2} \longrightarrow \mathcal{P}_3$ $\quad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+d)x^2 + ax + (b-3c)$

$B_{\mathrm{Mat}_{2\times2}} = \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \}$ $\quad B_{\mathcal{P}_3} = \{1, x, x^2\}$

$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = x^2 + x \rightsquigarrow [\ ]_{B_{\mathcal{P}_3}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$; $T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = -3 \rightsquigarrow [\ ]_{B_{\mathcal{P}_3}} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 1 \rightsquigarrow [\ ]_{B_{\mathcal{P}_3}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = x^2 \rightsquigarrow [\ ]_{B_{\mathcal{P}_3}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\rightsquigarrow [T]_{B_{\mathrm{Mat}_{2\times2}} B_{\mathcal{P}_3}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & -3 & 1 \end{bmatrix}$

Algebraic Properties

3 operations on $T: V \longrightarrow W$.

(I) ADDITION

$F+G: V \longrightarrow W$ linear

$$(F+G)(\vec{v}) = F(\vec{v}) \underset{\text{in } W}{+} G(\vec{v})$$

$$[F+G] = [F] + [G]$$

Using bases B_V, B_W

(same for all 3 matrices)

(II) SCALAR MULTIPLICATION

$\alpha T: V \longrightarrow W$ linear

$$(\alpha T)(\vec{v}) = \alpha \underset{\text{in } W}{\cdot} T(\vec{v})$$

$$[\alpha T] = \alpha [T]$$

Using bases B_V, B_W

(same for both matrices)

(III) COMPOSITION

$F: V \longrightarrow W$ linear

$G: W \longrightarrow U$ linear

Then $G \circ F: V \longrightarrow U$ linear

$$(G\circ F)(\vec{v}) = G(\underbrace{F(\vec{v})}_{\text{in } W})$$

$$\underset{m \times n}{[G\circ F]_{B_V B_U}} = \underset{m \times \ell}{[G]_{B_W B_U}} \; \underset{\ell \times n}{[F]_{B_V B_W}}$$

$\dim V = n$

$\dim W = \ell$

$\dim U = m$

Any B_W.

Application $T: V \longrightarrow W$ is invertible if and only if $[T]_{B_V B_W}$ is invertible for any B_V = basis for V & B_W = basis for W (so $\dim V = \dim W$)

Example: $F : \mathrm{Mat}_{2\times 2} \longrightarrow \mathbb{P}_2$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to (a+d)x^2 + ax + (b-c)$$

$G : \mathbb{P}_2 \longrightarrow \mathbb{R}^2$

$$p(x) \longmapsto \begin{bmatrix} p(1) \\ p'(2) \end{bmatrix}$$