Lecture 27: §5.9 Matrix representations for linear transformations

Recall A linear transf $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is always given by a matrix $\underset{m\times n}{A}$

$$A = [T(\vec{e}_1), \dots, T(\vec{e}_n)]$$

Ex: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 - x_2 \\ 2x_1 + 4x_2 \\ x_2 \end{bmatrix} \qquad A = \left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right), T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right] = \begin{bmatrix} 1 & -1 \\ 2 & 4 \\ 0 & 1 \end{bmatrix}$$

TODAY'S GOAL: Find a similar way to interpret $T: V \longrightarrow W$ linear transf

Summary:
- If $\dim V = n$ $\quad B_V = \{\vec{v}_1, \dots, \vec{v}_n\}$ basis for V
- If $\dim W = m$ $\quad B_W = \{\vec{w}_1, \dots, \vec{w}_m\}$ basis for W

Then

$$\begin{tikzcd}
\vec{v} \arrow[d, mapsto] & V \arrow[r, "T"] \arrow[d, "\wr \text{ iso}"] & W & \vec{w} = b_1\vec{w}_1 + \dots + b_m\vec{w}_m \\
{[\vec{v}]_{B_V}} & \mathbb{R}^n \arrow[r, "\tilde{T}", "\text{matrix } A"'] & \mathbb{R}^m \arrow[u, "\wr"] & \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \arrow[u, mapsto]
\end{tikzcd}$$

$$\boxed{A = \left[[T(\vec{v}_1)]_{B_W} \cdots [T(\vec{v}_n)]_{B_W} \right]}$$

$$\rightsquigarrow \boxed{[T(\vec{v})]_{B_W} = A[\vec{v}]_{B_V}} \text{ in } \mathbb{R}^m$$

NAME: A = matrix of T relative to the bases B_V & B_W.

Three Fundamental Facts

$T: V \longrightarrow W$ linear

FACT 1 T is completely determined by its values on a basis for V.

FACT 2 If $\dim V = n$ & $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for V, then

$T: V \longrightarrow \mathbb{R}^n$ is linear and invertible with $T^{-1}: \mathbb{R}^n \longrightarrow V$

$$\vec{v} \longmapsto [\vec{v}]_B \qquad\qquad \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix} \longmapsto \alpha_1\vec{v}_1 + \dots + \alpha_n \vec{v}_n$$

Write $\boxed{T = T_B}$ to emphasize the crucial role of the basis B.

Upshot: By choosing a basis B for V we can always think of V as $\mathbb{R}^n$.

Conclusion 1: If we choose coordinates for V & W by picking a basis for each space, then T can be thought of as $\tilde{T}: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear

FACT 3 Composition of linear maps is linear

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[rr, bend right] & W \arrow[r, "G"] & U \end{tikzcd}$$

$G \circ F(\vec{v}) = G(F(\vec{v}))$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "(T_{B_V})^{-1}", "\text{linear}"'] \arrow[rrr, bend right, "\tilde{T}"'] & V \arrow[r, "T", "\text{linear}"'] & W \arrow[r, "T_{B_W}", "\text{linear}"'] & \mathbb{R}^m \end{tikzcd}$$

$\tilde{T}$ linear

matrix $= [T]_{B_V B_W}$

$\boxed{\tilde{T} = T_{B_V B_W}}$

Examples

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[d, "T_{B_W}"] \\ \mathbb{R}^n \arrow[u, "T_{B_V}^{-1}"] \arrow[r, "\tilde{T}"] & \mathbb{R}^m \end{tikzcd}$$

① $T : \mathcal{P}_3 \longrightarrow \mathcal{P}_2$ $\qquad T(a+bx+cx^2+dx^3) = b + 2cx + 3dx^2$

$P_{(x)} \longmapsto P'_{(x)}$

Choose $B_{\mathcal{P}_3} = \{1, x, x^2, x^3\}$ $\qquad T^{-1}_{B_{\mathcal{P}_3}} : \mathbb{R}^4 \longrightarrow \mathcal{P}_3$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longmapsto ax^3+bx^2+cx+d$$

$B_{\mathcal{P}_2} = \{1, x, x^2\}$ $\qquad T_{B_{\mathcal{P}_2}} : \mathcal{P}_2 \longrightarrow \mathbb{R}^3$

$$r+sx+tx^2 \longmapsto \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$$T_{B_{\mathcal{P}_2}B_{\mathcal{P}_3}} : \mathbb{R}^4 \xrightarrow{T^{-1}_{B_{\mathcal{P}_3}}} \mathcal{P}_3 \xrightarrow{T} \mathcal{P}_2 \xrightarrow{T_{B_{\mathcal{P}_2}}} \mathbb{R}^3$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longmapsto \underset{a+bx+cx^2+dx^3}{\overset{\parallel}{P}} \longrightarrow \underset{\underbrace{b}_{=r}+\underbrace{2c}_{=s}x+\underbrace{3d}_{t}x^2}{\overset{\parallel}{P'}} \longrightarrow \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}$$

Conclude: $F = T_{B_{\mathcal{P}_2}B_{\mathcal{P}_2}}\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}$ linear map $\mathbb{R}^4 \longrightarrow \mathbb{R}^3$

$$\text{Matrix} = \begin{bmatrix} F(\vec{e}_1) & F(\vec{e}_2) & F(\vec{e}_3) & F(\vec{e}_4) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad 3\times 4$$

$[T]_{B_{\mathcal{P}_2}B_{\mathcal{P}_3}}$ $\qquad F(\vec{e}_1) = [T(1)]_{B_{\mathcal{P}_2}}$ $\qquad F(\vec{e}_2) = [T(x)]_{B_{\mathcal{P}_2}}$ $\qquad F(\vec{e}_3) = [T(x^2)]_{B_{\mathcal{P}_2}}$ $\qquad F(\vec{e}_4) = [T(x^3)]_{B_{\mathcal{P}_2}}$

② $T: \mathrm{Mat}_{2\times3} \longrightarrow \mathbb{R}^3$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A \longmapsto \begin{bmatrix} 2a_{11} - 4a_{22} \\ a_{13} + a_{21} \\ a_{12} - a_{23} \end{bmatrix}$$

$B_{\mathrm{Mat}_{2\times3}} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$

$B_{\mathbb{R}^3} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\tilde{T} = T_{B_{\mathrm{Mat}_{2\times3}} B_{\mathbb{R}^3}} : \mathbb{R}^6 \xrightarrow{(T_{B_{\mathrm{Mat}}})^{-1}} \mathrm{Mat}_{2\times3} \xrightarrow{T} \mathbb{R}^3 \xrightarrow[\underline{\underline{id}}]{T_{B_{\mathbb{R}^3}}} \mathbb{R}^3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix} \longrightarrow \begin{bmatrix} 2x_1 - 4x_5 \\ x_3 + x_4 \\ x_2 - x_6 \end{bmatrix} \longrightarrow \begin{bmatrix} 2x_1 - 4x_5 \\ x_3 + x_4 \\ x_2 - x_6 \end{bmatrix}$$

$$\text{Matrix for } \tilde{T} = \begin{bmatrix} \tilde{T}(\vec{e_1}), \cdots \tilde{T}(\vec{e_6}) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \quad 3\times6$$

$$\tilde{T}(\vec{e_1}) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \longleftrightarrow T\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\& \quad [\vec{v}]_{B_{\mathbb{R}^3}} = \vec{v}$$

$(B_{\mathbb{R}^3} = \text{standard basis})$

$$\tilde{T}(\vec{e_2}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \longleftrightarrow T\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{T}(\vec{e_3}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \tilde{T}(\vec{e_4}) \longleftrightarrow T\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right)$$

$$\tilde{T}(\vec{e_5}) = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix} \longleftrightarrow T\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{T}(\vec{e_6}) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \longleftrightarrow T\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$T: \mathrm{Mat}_{2\times 3} \longrightarrow \mathbb{R}^3 \qquad B_{\mathrm{Mat}_{2\times 3}} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A \longmapsto \begin{bmatrix} 2a_{11} - 4a_{22} \\ a_{13} + a_{21} \\ a_{12} - a_{23} \end{bmatrix}$$

Q: What if we chose a different basis for $\mathbb{R}^3$?

$$B'_{\mathbb{R}_3} = \left\{ \overset{\vec{w}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{\vec{w}_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \overset{\vec{w}_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \right\} \qquad \text{New } A = \begin{bmatrix} 2 & 0 & -1 & -1 & -4 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\tilde{T} := T_{B_{\mathrm{Mat}_{2\times 3}} B'_{\mathbb{R}^3}} : \mathbb{R}^6 \xrightarrow{(T_{B_{\mathrm{Mat}}})^{-1}} \mathrm{Mat}_{2\times 3} \xrightarrow{T} \mathbb{R}^3 \xrightarrow{T_{B'_{\mathbb{R}^3}}} \mathbb{R}^3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix} \longrightarrow \begin{bmatrix} 2x_1 - 4x_5 \\ x_3 + x_4 \\ x_2 - x_6 \end{bmatrix} \longrightarrow \left[\begin{bmatrix} 2x_1 - 4x_5 \\ x_3 + x_4 \\ x_2 - x_6 \end{bmatrix}\right]_{B'_{\mathbb{R}^3}} = \begin{bmatrix} 2x_1 - x_3 - x_4 - 4x_5 \\ -x_2 + x_3 + x_4 + x_6 \\ x_2 - x_6 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longrightarrow \begin{bmatrix} a-b \\ b-c \\ c \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\vec{w}_1 \rightsquigarrow [\ ]_{B'} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{w}_3 - \vec{w}_2 = -\vec{w}_2 + \vec{w}_3 \rightsquigarrow [\ ]_{B'} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{w}_2 - \vec{w}_1 \rightsquigarrow [\ ]_{B'} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix} = -4\vec{w}_1 \rightsquigarrow [\ ]_{B'} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = -(\vec{w}_3 - \vec{w}_2) = \vec{w}_2 - \vec{w}_3 \rightsquigarrow [\ ]_{B'} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Matrix Representation

Representation Theorem: Fix $T: V \longrightarrow W$ linear transf with $\dim V = n$ & $\dim W = m$. Pick $B_V = \{\vec{v}_1, \dots, \vec{v}_n\}$ basis for V

$B_W = \{\vec{w}_1, \dots, \vec{w}_m\}$ —— W.

Then, T, B_V & B_W gives rise to a linear transformation

$$T_{B_V B_W} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

defined by $T_{B_V B_W}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = T_{B_W} \circ T \circ T_{B_V}^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right)$

$$= \left[T(a_1\vec{v}_1 + \dots + a_n\vec{v}_n)\right]_{B_W}$$

Furthermore, the associated $m \times n$ matrix is obtained as follows:

$$\boxed{[T]_{B_V B_W}} = \left[\, [T(\vec{v}_1)]_{B_W}, \dots, [T(\vec{v}_n)]_{B_W} \,\right]$$

Consequence: $T(\vec{v}) = \beta_1 \vec{w}_1 + \dots + \beta_m \vec{w}_m$ where $\underset{m\times 1}{\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}} = \underset{m\times n}{[T]_{B_V B_W}} \underset{n \times 1}{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}$

$\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n$

Examples

① $T: \mathcal{P}_3 \longrightarrow \mathcal{P}_2$, $\quad P(x) \longmapsto P'(x)$

$B_{\mathcal{P}_3} = \{1, x, x^2, x^3\}$

$B_{\mathcal{P}_2} = \{1, x, x^2\}$

Q $[T]_{B_{\mathcal{P}_3} B_{\mathcal{P}_2}} = ?$

A: Matrix $= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$T(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \rightsquigarrow [T(1)]_{B_{\mathcal{P}_2}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$T(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \rightsquigarrow [T(x)]_{B_{\mathcal{P}_2}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 \rightsquigarrow [T(x^2)]_{B_{\mathcal{P}_2}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$T(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 \rightsquigarrow [T(x^3)]_{B_{\mathcal{P}_2}} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

So $T(a1 + bx + cx^2 + dx^3) = b \cdot 1 + 2c \cdot x + 3d x^2$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{= [P]_{\mathcal{P}_3}} = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix} = [T(P)]_{B_{\mathcal{P}_2}}$$

② $T: \mathcal{P}_2 \longrightarrow \mathbb{R}^2$ $\qquad B_{\mathcal{P}_2} = \{1, x, x^2\}$

$a+bx+cx^2 \mapsto \begin{bmatrix} c-b \\ 2c+a \end{bmatrix}$ $\qquad B_{\mathbb{R}^2} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{w}_1}, \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{w}_2} \right\}$

$T(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{w}_1 + \vec{w}_2 \rightsquigarrow [T(1)]_{B_{\mathbb{R}^2}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$T(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\vec{w}_1 \rightsquigarrow [T(x)]_{B_{\mathbb{R}^2}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\qquad \rightsquigarrow [T]_{B_{\mathcal{P}_2} B_{\mathbb{R}^2}} = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

$T(x^2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\vec{w}_2 + 3\vec{w}_1 \rightsquigarrow [T(x^2)]_{B_{\mathbb{R}^2}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

- $[T(3+4x+5x^2)]_{B_{\mathbb{R}^2}} = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3-4+15 \\ 3+10 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \end{bmatrix}$

So $T(3+4x+5x^2) = 14\vec{w}_1 + 13\vec{w}_2 = 14\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 13\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 13 \end{bmatrix}$.

③ $T: \mathrm{Mat}_{2\times 2} \longrightarrow \mathcal{P}_3$ $\qquad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+d)x^2 + ax + (b-3c)$

$B_{\mathrm{Mat}_{2\times2}} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ $\qquad B_{\mathcal{P}_3} = \{1, x, x^2\}$

$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = x^2 + x \rightsquigarrow [\;]_{B_{\mathcal{P}_3}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$; $\quad T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = -3 \rightsquigarrow [\;]_{B_{\mathcal{P}_3}} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 1 \rightsquigarrow [\;]_{B_{\mathcal{P}_3}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\qquad T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = x^2 \rightsquigarrow [\;]_{B_{\mathcal{P}_3}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\rightsquigarrow [T]_{B_{\mathrm{Mat}_{2\times2}} B_{\mathcal{P}_3}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & -3 & 1 \end{bmatrix}$

Algebraic Properties

3 operations on $T: \mathbb{V} \to \mathbb{W}$.

(I) ADDITION	(II) SCALAR MULTIPLICATION	(III) COMPOSITION
$F+G: \mathbb{V} \to \mathbb{W}$ linear $(F+G)(\vec{v}) = F(\vec{v}) + \underset{\text{in } \mathbb{W}}{G(\vec{v})}$	$\alpha T: \mathbb{V} \to \mathbb{W}$ linear $(\alpha T)(\vec{v}) = \alpha \cdot \underset{\text{in } \mathbb{W}}{T(\vec{v})}$	$F: \mathbb{V} \to \mathbb{W}$ linear $G: \mathbb{W} \to \mathbb{U}$ linear Then $G \circ F: \mathbb{V} \to \mathbb{U}$ linear $(G \circ F)(\vec{v}) = G(\underbrace{F(\vec{v})}_{\text{in } \mathbb{W}})$
$[F+G] = [F] + [G]$ Using bases $B_{\mathbb{V}}$, $B_{\mathbb{W}}$ (same for all 3 matrices)	$[\alpha T] = \alpha [T]$ Using bases $B_{\mathbb{V}}$, $B_{\mathbb{W}}$ (same for both matrices)	$[G \circ F]_{B_{\mathbb{V}} B_{\mathbb{U}}} = [G]_{B_{\mathbb{W}} B_{\mathbb{U}}} [F]_{B_{\mathbb{V}} B_{\mathbb{W}}}$ $m \times n$ $\quad$ $m \times \ell$ $\quad$ $\ell \times n$ $\dim \mathbb{V} = n$ $\dim \mathbb{W} = \ell$ $\quad$ Any $B_{\mathbb{W}}$. $\dim \mathbb{U} = m$

Application $T: \mathbb{V} \to \mathbb{W}$ is invertible if and only if $[T]_{B_{\mathbb{V}} B_{\mathbb{W}}}$ is invertible for any $B_{\mathbb{V}}$ = basis for $\mathbb{V}$ & $B_{\mathbb{W}}$ = basis for $\mathbb{W}$ (so $\dim \mathbb{V} = \dim \mathbb{W}$)

Example: $F: Mat_{2\times 2} \longrightarrow \mathcal{P}_2$ $\qquad$ $G: \mathcal{P}_2 \longrightarrow \mathbb{R}^2$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to (a+d)x^2 + ax + (b-c) \qquad\qquad p_{(x)} \longmapsto \begin{bmatrix} p(1) \\ p'(2) \end{bmatrix}$$

$$(G\circ F)_{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = G\Big(\underbrace{(a+d)x^2 + ax + (b-c)}_{=p_{(x)}}\Big) = \begin{bmatrix} a+d+a+b-c \\ (2(a+d)x+a)\big|_2 \end{bmatrix} = \begin{bmatrix} 2a+b-c+d \\ 5a+4d \end{bmatrix}$$

$$B_{Mat_{2\times 2}} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \quad B_{\mathcal{P}_2} = \{1, x, x^2\}, \quad B_{\mathbb{R}^2} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$F\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = x^2 + x \rightsquigarrow [\ ]_{B_{\mathcal{P}_2}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad F\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = -1 \rightsquigarrow [\ ]_{B_{\mathcal{P}_2}} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$F\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 1 \rightsquigarrow [\ ]_{B_{\mathcal{P}_2}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad F\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = x^2 \rightsquigarrow [\ ]_{B_{\mathcal{P}_2}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$G(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad G(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad G(x^2) = \begin{bmatrix} 1 \\ 2\cdot 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$[F]_{B_{Mat_{2\times 2}} B_{\mathcal{P}_2}} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \qquad [G]_{B_{\mathcal{P}_2} B_{\mathbb{R}^2}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\rightsquigarrow [G\circ F]_{B_{Mat_{2\times 2}} B_{\mathcal{P}_2}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 5 & 0 & 0 & 4 \end{bmatrix} \checkmark$$

$$(G\circ F)\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad (G\circ F)\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (G\circ F)\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad (G\circ F)\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$