# Lecture 28: §6.2 Determinants

**Q:** What are determinants?

**A:** For each square matrix $A$ (# rows = # cols) we will define a number $\det(A)$

- $\det(A)$ is polynomial in the entries of $A$ (sums of $(\pm 1)\cdot$products of $a_{ij}$'s)
- Main properties:

① $A$ is singular ($N(A) \neq \{\vec{0}\}$) if, and only if, $\det(A) = 0$.

② det is multiplicative: $\det(AB) = \det(A)\det(B)$ $\quad A, B$ $n\times n$ matrices

③ det is compatible with elementary row operations (precise rules!)

④ If $A$ is upper triangular, ie $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \ddots & & \vdots \\ & 0 & & a_{nn} \end{bmatrix}$, then

$\det A = a_{11} \cdots a_{nn}$ = product of diag entries

→ zeros below the diagonal

⑤ $\det(A^T) = \det(A)$ ("row vs column expansion")

## The 2x2 case

Recursive definition for det(A) starts with 2x2 matrices

Fix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ 2x2 matrix $\rightsquigarrow$ $\boxed{\det(A) = a_{11}a_{22} - a_{12}a_{21}}$

(polynomial in entries!)

We check the 5 properties from slides 1

1. A singular if, and only if $\det(A) = 0$ (Lecture 8)
2. det is multiplicative

4. $\det\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = ad$
5. $\det(A) = \det(A^T)$
3. det & Row operations : next time.

# Formulas for larger matrices

**Recursive approach:** Define $\det(A)$ for $n\times n$ matrix in terms of determinants of submatrices of size $(n-1)\times(n-1)$, obtained by removing 1 row & 1 col from $A$ (like we did when defining $\vec{u}\times\vec{v}$ in $\mathbb{R}^3$)

**Def:** Cofactors of $A$ ($n\times n$ matrix) - Cofactor matrix

Given $1\leq r,s\leq n$ $\quad r=$ row label, $\quad s=$ column label, define:

- $M_{r,s} =$

- $\boxed{A_{r,s}} = (-1)^{r+s}\det(M_{r,s}) =$ $(r,s)$-cofactor of $A$ (signed minor)

- The cofactor matrix is $\boxed{\mathrm{Cof}(A)} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$

**Cofactor Formula:**

$$\boxed{A_{r,s}} = (-1)^{r+s} \det(M_{r,s}) \quad \& \quad \boxed{\det(A) = a_{11} A_{11} + a_{12} A_{12} + \cdots + a_{1n} A_{1n}}$$

**Example ①** $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $A_{1,1} = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} =$ ; $A_{1,2} = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} =$

$A_{2,1} = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} =$ ; $A_{2,2} = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} =$

$$\det(A) = a_{11} A_{11} + a_{12} A_{12}$$
$$= a_{11} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Q: How to number this?

$$\boxed{\det A} = a_{11} (+) \det \begin{bmatrix} a_{1} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$
$$+ \cdots + (-1)^{1+n} a_{1n} \det \begin{bmatrix} a_{11} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots \\ a_{n1} & & a_{n(n-1)} & a_{nn} \end{bmatrix}$$

$\boxed{A_{r,s}} = (-1)^{r+s} \det(M_{r,s})$ & $\boxed{\det(A) = a_{11} A_{11} + a_{12} A_{12} + \cdots + a_{1n} A_{1n}}$

Example 2: $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$ $\rightsquigarrow$ Cof(A) has 9 entries

(1,1) $A_{11} = \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$; (1,2) $A_{12} = \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$; (1,3) $A_{13} = \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$;

(2,1) $A_{2,1} = \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$; (2,2) $A_{2,2} = \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$; (2,3) $A_{2,3} = \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$;

(3,1) $A_{3,1} = \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$; (3,2) $A_{3,2} = \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$; (3,3) $A_{3,3} = \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$;

$\boxed{A_{r,s}} = (-1)^{r+s} \det(M_{r,s})$ & $\boxed{\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}}$

<u>Example 3</u>: Compute $\det(A)$ for $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & 2 & 3 & 1 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix}$

## Special case: Triangular matrices

**Def** $A$ of size $n\times n$ is triangular if it is lower or upper triangular

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \ddots & a_{(n-1)n} \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

upper triangular

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{bmatrix}$$

lower triangular

**Theorem:** If $A$ is triangular, then $\det(A) = a_{11}\cdots a_{nn}$
= product of diagonal entries.

Why?