Lecture 28: §6.2 Determinants

Q: What are determinants?

A: For each square matrix A (# rows = # cols) we will define a number $\det(A)$

- $\det(A)$ is polynomial in the entries of A (sums of $(\pm 1)\cdot$products of a_{ij}'s)
- Main properties:

① A is singular ($N(A) \neq \{\vec{0}\}$) if, and only if, $\det(A) = 0$.

② $\det$ is multiplicative: $\det(AB) = \det(A)\det(B)$ A, B $n\times n$ matrices

③ $\det$ is compatible with elementary row operations (precise rules!)

④ If A is upper triangular, i.e. $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \ddots & & \vdots \\ & 0 & & a_{nn} \end{bmatrix}$, then

$\det A = a_{11} \cdots a_{nn}$ = product of diag entries

zeros below the diagonal

Special case: $\det I_n = \det\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = (1)^n = 1.$

⑤ $\det(A^T) = \det(A)$ ("row vs column expansion")

The 2x2 case

Recursive definition for det(A) starts with 2x2 matrices

Fix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ 2x2 matrix $\rightsquigarrow$ $\boxed{\det(A) = a_{11}a_{22} - a_{12}a_{21}}$

(polynomial in entries!)

We check the 5 properties from slides 1

① A singular if, and only if $\det(A) = 0$ (Lecture 8)

② det is multiplicative

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} f & g \\ h & j \end{bmatrix}\right) = \det\begin{pmatrix} af+bh & ag+bj \\ cf+dh & cg+dj \end{pmatrix}$$

$$= (af+bh)(cg+dj) - (ag+bj)(cf+dh)$$

$$= afcg + afdj + bhcg + bdhj - agcf - adgh - bjcf - bjdh$$

$$= ad(fj-gh) - bc(fj-gh) = (ad-bc)(fj-gh) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\det\left(\begin{bmatrix} f & g \\ h & j \end{bmatrix}\right)$$

④ $\det\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = ad$

⑤ $\det(A) = \det(A^T)$: $\det\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = ad - cb = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

③ det & Row operations: next time.

Formulas for larger matrices

<u>Recursive approach</u>: Define $\det(A)$ for $n\times n$ matrix in terms of determinants of submatrices of size $(n-1)\times(n-1)$, obtained by removing 1 row & 1 col from A (like we did when defining $\vec{u}\times\vec{v}$ in $\mathbb{R}^3$)

<u>Def</u>: Cofactors of A ($n\times n$ matrix) - Cofactor matrix

Given $1\le r,s\le n$ r = row label, s = column label, define:

- $M_{r,s}$ = submatrix of A of size $(n-1)\times(n-1)$ obtained from A by removing row r & col s from A
- $\boxed{A_{r,s}} = (-1)^{r+s}\det(M_{r,s})$ = <u>(r,s)-cofactor of A</u> (signed minor)
- The <u>cofactor matrix</u> is $\boxed{\mathrm{Cof}(A)} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$

<u>Cofactor formula</u>: $\boxed{\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}}$

$$\boxed{A_{r,s}} = (-1)^{r+s} \det(M_{r,s}) \quad \& \quad \boxed{\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}}$$

Example ① $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$A_{1,1} = (-1)^{1+1} \det \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ \cancel{a_{21}} & a_{22} \end{bmatrix} = a_{22} \ ; \quad A_{1,2} = (-1)^{1+2} \det \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ a_{21} & \cancel{a_{22}} \end{bmatrix} = -a_{21}$$

$$A_{2,1} = (-1)^{1+2} \det \begin{bmatrix} \cancel{a_{11}} & a_{12} \\ \cancel{a_{21}} & \cancel{a_{22}} \end{bmatrix} = -a_{12} \ ; \quad A_{2,2} = (-1)^{2+2} \det \begin{bmatrix} a_{11} & \cancel{a_{12}} \\ \cancel{a_{21}} & \cancel{a_{22}} \end{bmatrix} = a_{11}$$

$$\begin{aligned} \det(A) &= a_{11}A_{11} + a_{12}A_{12} \\ &= a_{11} \det \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ \cancel{a_{21}} & a_{22} \end{bmatrix} - a_{12} \det \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ a_{21} & \cancel{a_{22}} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21} \ ✓ \end{aligned}$$

Obs: $\operatorname{Cof}(A) = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix} = \det(A)\,(A^{-1})^T$ (if A is invertible)

(This will be a general rule for any invertible $n \times n$ matrix: $A^{-1} = \frac{1}{\det A} (\operatorname{Cof} A)^T$!

Q: How to remember this?

$$\boxed{\det A} = a_{11}(+) \det \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cdots & \cancel{a_{1n}} \\ \cancel{a_{21}} & a_{22} & \cdots & a_{2n} \\ \cancel{a_{n1}} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - a_{12} \det \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} & \cdots & \cancel{a_{1n}} \\ a_{21} & \cancel{a_{22}} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & \cancel{a_{n2}} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$+ \cdots + (-1)^{1+n} a_{1n} \det \begin{bmatrix} \cancel{a_{11}} & \cdots & \cancel{a_{1(n-1)}} & \cancel{a_{1n}} \\ a_{21} & \cdots & a_{2(n-1)} & \cancel{a_{2n}} \\ \vdots & & & \\ a_{n1} & & a_{n(n-1)} & \cancel{a_{nn}} \end{bmatrix}$$

(start with + & alternate signs!)

$\boxed{A_{r,s}} = (-1)^{r+s} \det(M_{r,s})$ & $\boxed{\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}}$

Example ②: $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$ ⇝ $\mathrm{Cof}(A)$ has 9 entries $= \begin{bmatrix} 1 & -14 & -4 \\ -2 & -1 & 8 \\ -7 & 11 & -1 \end{bmatrix}$

(1,1) $A_{11} = (-1)^{1+1} \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = + \cdot 1$;

(1,2) $A_{12} = (-1)^{1+2} \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = -(2+12) = -14$;

(1,3) $A_{13} = (-1)^{1+3} \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = +(-4) = -4$;

(2,1) $A_{21} = (-1)^{2+1} \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = -2$;

(2,2) $A_{22} = (-1)^{2+2} \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = +(3-4) = -1$;

(2,3) $A_{23} = (-1)^{2+3} \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = -(-2 \cdot 4) = 8$;

(3,1) $A_{31} = (-1)^{3+1} \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = +(2(-3)-1) = -7$;

(3,2) $A_{32} = (-1)^{3+2} \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = -(-9-2) = 11$;

(3,3) $A_{33} = (-1)^{3+3} \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = +(3-4) = -1$;

$\det A = 3 \cdot 1 + 2 \cdot (-14) + 1 \cdot (-4) = 3 - 28 - 4 = \boxed{-29}$

• Check: $(\mathrm{Cof}\, A)^T A = (-29) I_3$

$$\boxed{A_{r,s}} = (-1)^{r+s}\det(M_{r,s}) \quad \& \quad \boxed{\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}}$$

Example ③: Compute $\det(A)$ for $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & 2 & 3 & 1 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix}$ $\rightsquigarrow$ Need A_{11}, A_{12}, A_{14} ($a_{13}=0$ so we don't need A_{13})

$\underline{A}$: $\det(A) = 1(-15) + 2\cdot(-18) + 2\cdot(-6) = \boxed{-63}$

- $A_{11} = (-1)^{1+1}\det\left(\begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & 2 & 3 & 1 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & -2 & 1 \end{bmatrix}\right)$

$$= 2\det\left(\begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & -2 & 1 \end{bmatrix}\right) - 3\det\left(\begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & -2 & 1 \end{bmatrix}\right) + 1\det\left(\begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & -2 & 1 \end{bmatrix}\right)$$

$$= 2(-1) - 3\cdot 2 + 1(-4-3) = -2-6-7 = -15$$

- $A_{12} = (-1)^{1+2}\det\left(\begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & 2 & 3 & 1 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix}\right) = -\det\left(\begin{bmatrix} -1 & 3 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix}\right)$

$$= -\left(-1\det\left(\begin{bmatrix} -1 & 3 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix}\right) - 3\det\left(\begin{bmatrix} -1 & 3 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix}\right) + 1\det\left(\begin{bmatrix} -1 & 3 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix}\right)\right)$$

$$= -\left(-(-1) - 3(-3) + (6+2)\right) = -(1+9+8) = -18.$$

- $A_{14} = (-1)^{1+4}\det\left(\begin{bmatrix} -1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -3 & -2 \end{bmatrix}\right) = -\left(-1\det\begin{pmatrix} 2 & -1 \\ -3 & -2 \end{pmatrix} - 2\det\begin{pmatrix} -3 & -1 \\ 2 & -2 \end{pmatrix} + 3\det\begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}\right) = -6$

Special case: Triangular matrices

Def A of size $n \times n$ is triangular if it is lower or upper triangular

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & a_{(n-1)n} \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

zeros below the diag.

upper triangular

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{bmatrix}$$

zeros above the diag.

lower triangular

Theorem: If A is triangular, then $\det(A) = a_{11} \cdots a_{nn}$
= product of diagonal entries.

Why? Assume A is lower triangular.

$$\det\left(\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{bmatrix}\right) = a_{11}(-1)^{1+1} \det\left(\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{bmatrix}\right) + 0$$

$$= a_{11}\left(a_{22}(-1)^{1+1} \det\begin{pmatrix} a_{22} & 0 & 0 \\ a_{32} & \ddots & 0 \\ \vdots & \cdots & a_{nn} \end{pmatrix} + 0\right)$$

Keep going until we get $a_{11} a_{22} \cdots a_{nn}$

If A is upper triangular: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$

• Option 1: Use $\det(A^T) = \det(A)$ (we have not yet seen why!)

then $A^T = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{12} & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ a_{1n} & \cdots & & a_{nn} \end{pmatrix}$ lower triangular so

$\det(A) = \det(A^T) = a_{11} \cdots a_{nn}$

• Option 2: Write the cofactor formula & use $\det(A) = 0$ for singular matrices.

$$\det(A) = a_{11} \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots 0 & a_{nn} \end{bmatrix} - \underbrace{a_{12} \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & & \\ \vdots & & & \\ 0 & 0 & & a_{nn} \end{bmatrix} + \cdots}$$

$$\underbrace{+ (-1)^{1+n} a_{1n} \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots 0 & a_{nn} \end{bmatrix}}$$

↳ all these matrices will be singular because their 1st column is $= 0$, so dets $= 0$

$$= a_{11} \det \begin{bmatrix} a_{22} & \cdots & a_{2n} \\ 0 & \ddots & \vdots \\ 0 & \cdots 0 & a_{nn} \end{bmatrix} = a_{11} a_{22} \det \begin{bmatrix} a_{33} & \cdots & a_{3n} \\ 0 & \ddots & \\ 0 & \cdots 0 & a_{nn} \end{bmatrix} = \cdots = a_{11} a_{22} \cdots a_{nn}.$$