Lecture 29: §6.3 Elementary operations & Determinants

Last time ① Defined determinants via cofactor formula.

- Cofactors of A = signed determinants of submatrices of A with 1 less row & col.

  ↳ $A_{r,s} = (-1)^{r+s} \cdot \det(A \text{ without row } r \text{ \& column } s)$

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & & a_{nn} \end{pmatrix} \rightsquigarrow \boxed{\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}}$

- 2×2 case: $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Example $\det\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} =$

② For Triangular matrices $A = \begin{pmatrix} a_{11} & & a_{1n} \\ 0 & \ddots & \\ 0 \cdots 0 & & a_{nn} \end{pmatrix}$ or $\begin{pmatrix} a_{11} & 0 \cdots & 0 \\ & \ddots & 0 \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ zeros above the diagonal

zeros below diag

$\rightsquigarrow \det(A) = \text{product of diagonal entries} = a_{11}a_{22}\cdots a_{nn}$.

TODAY: Use elementary row operations to simplify calculation of determinants.

$\boxed{\text{3 row operations} \longleftrightarrow \text{3 effects on determinants}}$

<u>INPUT</u>: $n \times n$ matrix A $\xrightarrow{\text{row op.}}$ <u>OUTPUT</u>: $n \times n$ matrix B

<u>Q</u>: How are $\det(B)$ & $\det(A)$ related?

<u>A</u>:

Operation	$\det(B)$	Net Effect
(I) $R_i \leftrightarrow R_j$, $i \neq j$ (exchange 2 rows)		
(II) $R_i \to \alpha R_i$, $\alpha \neq 0$ scalar (multiply a row by a <u>nonzero</u> scalar)		
(III) $R_i \to R_i + \beta R_j$, $i \neq j$ (Add to a row a scalar multiple of a <u>different</u> row)		

Operation I : Exchange 2 rows

$A \xrightarrow[R_i \leftrightarrow R_j]{} B$

Effect: $\det(B) = -\det(A)$.

- 2×2 case

Example: ① $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_1 \longleftrightarrow R_2]{} B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 3 & 4 & 5 \end{pmatrix}$

② $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_2 \longleftrightarrow R_3]{} B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 1 & 1 & 2 \end{pmatrix}$

Operation II: Multiply a row by a nonzero scalar α

$A \longrightarrow B$

$R_i \to \alpha R_i$

$\alpha \neq 0$

Effect: $\det(B) = \alpha \cdot \det(A)$

Example: ① $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_1 \to \alpha R]{} B = \begin{pmatrix} \alpha & 0 & 2\alpha \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}$

② $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_1 \to \alpha R]{} B = \begin{pmatrix} 1 & 0 & 2 \\ \alpha & \alpha & 2\alpha \\ 3 & 4 & 5 \end{pmatrix}$

$\boxed{\text{Operation III: Add to a row a scalar multiple of a different row}}$ $A \xrightarrow[R_i \to R_i + \beta R_j,\ i \neq j]{} B$

Effect: $\det(B) = \det(A)$

Example: ① $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 4 & 5 \end{pmatrix}$

② $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_1 \to R_1 + R_2]{} B = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}$

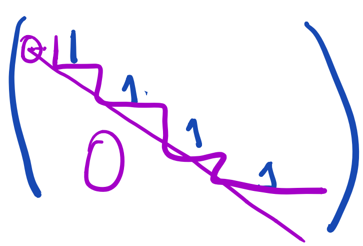
Combine all 3 operations

ALGORITHM for computing $\det(A)$

1. Use row operations to go from A to B with B in EF (keeping a record of them!)
2. Compute $\det(B)$
3. Trace back the operations $A \longrightarrow B$ & use the rules to get $\det(A)$ from $\det(B)$.

Key: A matrix in EF is upper triangular, so $\det B$ is easy to compute (just take the product of diag entries)

Why? Each step start either on or to the right of the diag



Ex: $\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ & $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

<u>Example</u>:

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 4 & 8 & 4 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix}$$

$\boxed{\text{Determinants to test for singular matrices}}$

Theorem: A of size $n \times n$ is invertible if, and only if, $\det A \neq 0$.