Lecture 29 : §6.3 Elementary operations & Determinants

Last time ① Defined determinants via cofactor formula.

- Cofactors of A = signed determinants of submatrices of A with 1 less row & col.

  ↳ $A_{r,s} = (-1)^{r+s} \cdot \det(A \text{ without row } r \text{ \& column } s)$

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & & a_{nn} \end{pmatrix} \rightsquigarrow \boxed{\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}}$

- 2x2 case: $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Example $\det\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} = \underset{+}{1 \cdot (-1)^{1+1}} \det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} + \underset{-}{0 \cdot (-1)^{1+2}} \det\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} + \underset{+}{2 (-1)^{1+3}} \det\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$

$= 1(5-8) - 0 + 2(4-3) = -3+2 = 1$

② For Triangular matrices $A = \begin{pmatrix} a_{11} & & a_{1n} \\ 0 & \ddots & \\ 0 - 0 & & a_{nn} \end{pmatrix}$ or $\begin{pmatrix} a_{11} & 0 \cdots 0 \\ & \ddots \; 0 \\ a_{n1} & \cdots \; a_{nn} \end{pmatrix}$ ↝ zeros above the diagonal

↝ zeros below diag

$\rightsquigarrow \det(A) =$ product of diagonal entries $= a_{11}a_{22}\cdots a_{nn}$.

TODAY: Use elementary row operations to simplify calculation of determinants.

$\boxed{\text{3 row operations} \longleftrightarrow \text{3 effects on determinants}}$

INPUT: $n \times n$ matrix A $\xrightarrow{\text{row op.}}$ OUTPUT: $n \times n$ matrix B

Q: How are $\det(B)$ & $\det(A)$ related?

A:

Operation	$\det(B)$	Net Effect
(I) $R_i \leftrightarrow R_j$, $i \neq j$ (exchange 2 rows)	$-\det(A)$	sign switch
(II) $R_i \to \alpha R_i$, $\alpha \neq 0$ scalar (multiply a row by a nonzero scalar)	$\alpha \det(A)$	multiply by scalar α
(III) $R_i \to R_i + \beta R_j$, $i \neq j$ (Add to a row a scalar multiple of a different row)	$\det(A)$	No change

Operation I : Exchange 2 rows

$$A \xrightarrow[R_i \leftrightarrow R_j]{} B$$

Effect: $\det(B) = -\det(A)$.

- 2x2 case $\quad \det\begin{pmatrix} a & b \\ c & d\end{pmatrix} = ad - bc \quad$ & $\quad \det\begin{pmatrix} c & d \\ a & b\end{pmatrix} = cb - ad = -\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}$.

Example: ① $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5\end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 3 & 4 & 5\end{pmatrix}$ $\quad \det B = -1 = \det A$.

$$\det(A) = +\det\begin{pmatrix} 1 & 2 \\ 4 & 5\end{pmatrix} + 2\det\begin{pmatrix} 1 & 1 \\ 3 & 4\end{pmatrix} = (5-8) + 2(4-3) = -3+2 = -1$$

$$\det(B) = \det\begin{pmatrix} 0 & 2 \\ 4 & 5\end{pmatrix} - \det\begin{pmatrix} 1 & 2 \\ 3 & 5\end{pmatrix} + 2\det\begin{pmatrix} 1 & 0 \\ 3 & 4\end{pmatrix} = (0-8) - (5-6) + 2(4-0)$$
$$= -8+1+8 = 1$$

② $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5\end{pmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{} B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 1 & 1 & 2\end{pmatrix}$ $\quad \det B = 1\,\underbrace{\det\begin{pmatrix} 4 & 5 \\ 1 & 2\end{pmatrix}}_{} - 0 + 2\,\underbrace{\det\begin{pmatrix} 3 & 4 \\ 1 & 2\end{pmatrix}}_{} = -1$

← sign gained here ↑

Reason behind the rule: ① Assume $\det(A^T) = \det(A)$.

② check the same rule applies when exchanging 2 columns in A (ie sign change)

③ use $\det(A) = a_{11}\underbrace{\det(M_{1,1})}_{} - a_{12}\underbrace{(\det M_{1,2})}_{} \cdots + (-1)^{1+n} a_{1n}\underbrace{\det(M_{1,j})}_{}$

sign changes in all terms (smaller size matrix)

Operation II: Multiply a row by a nonzero scalar α

$A \longrightarrow B$

$R_i \to \alpha R_i$

$\alpha \neq 0$

Effect: $\det(B) = \alpha \cdot \det(A)$

- **2x2 case** ($i=1$):

$$\det\begin{pmatrix} \alpha a & \alpha b \\ c & d \end{pmatrix} = \alpha ad - \alpha bc = \alpha(ad-bc) = \alpha \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det\begin{pmatrix} a & b \\ \alpha c & \alpha d \end{pmatrix} = a\alpha d - b\alpha c = \alpha(ad-bc) = \alpha \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Example: ① $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_1 \to \alpha R]{} B = \begin{pmatrix} \alpha & 0 & 2\alpha \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}$

$$\det(B) = \alpha \det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} - 0 + 2\alpha \det\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$$

$$= \alpha\Big(\underbrace{1 \det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} + 2 \det\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}}_{=\det(A)}\Big) = \alpha(-1) = \alpha \det(A)$$

② $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_1 \to \alpha R]{} B = \begin{pmatrix} 1 & 0 & 2 \\ \alpha & \alpha & 2\alpha \\ 3 & 4 & 5 \end{pmatrix}$

$$\det B = 1 \underbrace{\det\begin{pmatrix} \alpha & 2\alpha \\ 4 & 5 \end{pmatrix}}_{=\alpha \det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}} - 0 + 2 \underbrace{\det\begin{pmatrix} \alpha & \alpha \\ 3 & 4 \end{pmatrix}}_{=\alpha\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}} = \alpha \det(A)$$

Reasons for rule: ① If multiply 1^{st} row: we scale a_{1j}'s in cofactor formula

② ________ another row: ________ cofactors A_{ij}.

Operation III: Add to a row a scalar multiple of a different row

$A \xrightarrow[\substack{R_i \to R_i + \beta R_j \\ i \neq j}]{} B$

Effect: $\det(B) = \det(A)$

Example: ① $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 4 & 5 \end{pmatrix}$

$\det(B) = 1 \det\begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix} + 2 \det\begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} = 1(5) + 2(-3) = -1 = \det(A)$

② $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_1 \to R_1 + R_2]{} B = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}$

$\det(B) = 2 \det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} - 1 \det\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} + 4 \det\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$

$= 2(5-8) - (5-6) + 4(4-3) = -6 + 1 + 4 = -1 = \det(A)$

Reason behind the rule: Distributive Laws on $\mathbb{R}$.

$\det(B) = \det(A) + \det \begin{array}{c} i \\ \\ j \end{array} \begin{pmatrix} \alpha R_j \\ \vdots \\ R_j \end{pmatrix} = \det(A) + \alpha \underbrace{\overbrace{\det\begin{pmatrix} \text{2 repeated} \\ \text{rows} \end{pmatrix}}^{C}}_{=0}$

(Exchange the rows to get: $\det(C) = -\det(C) \leadsto \det C = 0$)

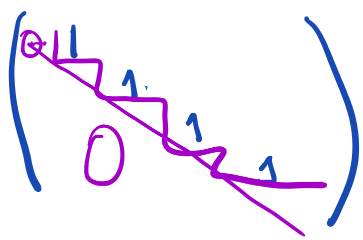
Combine all 3 operations

ALGORITHM for computing $\det(A)$

1. Use row operations to go from A to B with B in EF (keeping a record of them!)
2. Compute $\det(B)$
3. Trace back the operations $A \longrightarrow B$ & use the rules to get $\det(A)$ from $\det(B)$.

Key: A matrix in EF is upper triangular, so $\det B$ is easy to compute (just take the product of diag entries)

Why? Each step start either on or to the right of the diag



Ex: $\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ & $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Example:

$$A=\begin{bmatrix}1&2&0&2\\0&4&8&4\\-3&2&-1&0\\2&-3&-2&1\end{bmatrix}\xrightarrow[R_3\to R_3+3R_1]{\text{(III)}}\begin{bmatrix}1&2&0&2\\0&4&8&4\\0&8&-1&6\\2&-3&-2&1\end{bmatrix}\xrightarrow[R_4\to R_4-2R_1]{\text{(III)}}\begin{bmatrix}1&2&0&2\\0&4&8&4\\0&8&-1&6\\0&-7&-2&-3\end{bmatrix}$$

$\det A \qquad\qquad \det A \qquad\qquad \det A$

$$\xrightarrow[R_3\to R_3-2R_2]{\text{(III)}}\begin{bmatrix}1&2&0&2\\0&4&8&4\\0&0&-17&-2\\0&-7&-2&-3\end{bmatrix}\xrightarrow[R_2\to R_2/4]{\text{(II)}}\begin{bmatrix}1&2&0&2\\0&1&2&1\\0&0&-17&-2\\0&-7&-2&-3\end{bmatrix}\xrightarrow[R_4\to R_4+7R_2]{\text{(III)}}\begin{bmatrix}1&2&0&2\\0&1&2&1\\0&0&-17&-2\\0&0&12&4\end{bmatrix}$$

$\det A \qquad\qquad \frac{1}{4}\det A \qquad\qquad \frac{1}{4}\det A$

$$\xrightarrow[R_3\to \frac{-1}{17}R_3]{\text{(II)}}\begin{bmatrix}1&2&0&2\\0&1&2&1\\0&0&1&2/17\\0&0&12&4\end{bmatrix}\xrightarrow[R_4\to R_4-12R_3]{\text{(III)}}\begin{bmatrix}1&2&0&2\\0&1&2&1\\0&0&1&2/17\\0&0&0&44/17\end{bmatrix}\xrightarrow[R_4\to\frac{17}{44}R_4]{\text{(II)}}\boxed{\begin{bmatrix}1&2&0&2\\0&1&2&1\\0&0&1&2/17\\0&0&0&1\end{bmatrix}}$$

$\frac{-1}{17}\left(\frac{1}{4}\det A\right) \qquad\qquad \frac{-1}{68}\det A \qquad\qquad \frac{17}{44}\left(-\frac{\det A}{68}\right)$

$\square$ matrix is triangular (EF) so $\det \square = 1\cdot 1\cdot 1\cdot 1 = 1$

Conclusion: $\frac{17}{44}\left(-\frac{\det A}{68}\right)=1$ so $\det A = \frac{-68\cdot 44}{17} = -176$.

Determinants to test for singular matrices

Theorem: A of size $n\times n$ is invertible if, and only if, $\det A \neq 0$.

Why? Recall algorithm to build A^{-1} or show A is singular

$$(A \mid I_n) \overset{GJ}{\sim} (A' \mid B) \qquad A' \text{ in REF}$$

1. If A is invertible we get $A' = I_n$
2. If A is not invertible, then A' has a row of zeros.

- But G-J elimination says $\boxed{\det A' = \beta \det A}$ with $\beta \neq 0$

((I) $\beta = -1$, (II) $\beta = \alpha \neq 0$, (III) $\beta = 1$ $\rightsquigarrow$ Final β = product of intermediate ones)

① invertible case: $A' = I_n \rightsquigarrow \det A' = 1 \rightsquigarrow \det A = \frac{1}{\beta} \neq 0$

or

② singular case: A' has a row of zeros $\rightsquigarrow \det A' = 0 \rightsquigarrow \det A = \frac{0}{\beta} = 0$.

(REF $\rightsquigarrow$ triangular)

For the converse: $\det A = 0$ if and only if $\det A' = 0$.