Lecture 30 §6.4 Product Rule & Cramer's Rule

Recall: Row operations change determinants in specific ways. $A \longrightarrow B$

Operation	$\det(B)$	Net Effect
(I) $R_i \leftrightarrow R_j$, $i \neq j$ (exchange 2 rows)	$-\det(A)$	sign switch
(II) $R_i \to \alpha R_i$, $\alpha \neq 0$ scalar (multiply a row by a nonzero scalar)	$\alpha \det(A)$	multiply by scalar α
(III) $R_i \to R_i + \beta R_j$, $i \neq j$ (Add to a row a scalar multiple of a different row)	$\det(A)$	No change

- Assuming we know $\det(A^T) = \det(A)$, we can derive similar rules for analogous column operations on $A \longleftrightarrow$ row operations on A^T.

TODAY:
1. $\det(AB) = \det A \det B$ (Product Rule)
2. Cramer's Rule for solving $Ax = b$ with A invertible via determinants

$\boxed{\text{Product Rule}}$

Theorem 1: A, B of size $n \times n$, then $\det(AB) = \det(A) \cdot \det(B)$.
(We saw a proof for general 2×2 matrices in Lecture 28)

Consequence: If A invertible then $AA^{-1} = I_n$, so $\det(A)\det(A^{-1}) = 1$.
Then $\det A \neq 0$ & $\det(A^{-1}) = \frac{1}{\det A}$. Moreover, if $\det(A) = 0$ then A is singular.
(we knew this already)

Q: Why is the product formula valid in general?

• Special case: AB singular matrix. Then: A or B are singular (Lecture 9)
Conclude $\det(AB) = 0 = \det(A)\det(B)$. ($\det(\ ) = 0$ for the singular one).

Lemma: If A is singular of size $n \times n$, & B is any matrix of size $n \times n$, then AB is also singular.

. Let's check our claim (*) in an example.

<u>Example</u> $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$ $\leadsto$ $AB = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix}$

we write the operations needed to turn A into I_2 & check the same sequence of elementary row operations turns AB into B.

- <u>In general</u>: Need to see that $A \xrightarrow[op]{} A'$ gives $AB \xrightarrow[\text{same op.}]{} A'B$

for any of the 3 elementary row operations.

After this fact, we concatenate all operations and get $A' = I_n$ so $A'B = B$.

Cramer's Rule

GOAL Given a nonsingular $n \times n$ matrix A, we want to use determinants to find the unique solution to $A\underline{x} = \underline{b}$ for a fixed $\underline{b}$ in $\mathbb{R}^n$.

Theorem 2 (Cramer's Rule) Write $A = \underbrace{[A_1 \cdots A_n]}_{\text{(non-singular)}}$ & $\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

For each $i = 1, 2, \ldots, n$ we build a new matrix

$$B_i = [A_1 \cdots A_{i-1} \; b_i \; A_{i+1} \cdots A_n]$$ (replace i^{th} col of A with $\underline{b}$)

Then, the unique solution to $A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ satisfies:

$$x_1 = \frac{\det(B_1)}{\det(A)}, \quad x_2 = \frac{\det(B_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(B_n)}{\det(A)}.$$

Example: (1) Solve $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Cramer's Rule

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Theorem 2 (Cramer's Rule) Write $A = [A_1 \cdots A_n]$ (non-singular) & $\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

For each $i = 1, 2, \ldots, n$ we build a new matrix

$$B_i = [A_1 \cdots A_{i-1} \ b_i \ A_{i+1} \cdots A_n]$$ (replace i^{th} col of A with $\underline{b}$)

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Example: 2 Solve $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$A = [A_1 \cdots A_n]$ nonsingular

$A\underline{x} = \underline{b}$

Proof of Cramer's Rule

- We check $x_1 = \dfrac{\det([\underline{b}\ A_2 \cdots A_n])}{\det(A)}$. Others are similar.

So far:

$$\begin{cases} x_1 A_1^T = \underline{b}^T - x_2 A_2^T - \cdots - x_n A_n^T \\ \det \begin{bmatrix} x_1 A_1^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix} = x_1 \det(A) \end{cases}$$