Lecture 30 §6.4 Product Rule & Cramer's Rule

Recall: Row operations change determinants in specific ways. $A \longrightarrow B$

Operation	det(B)	Net Effect
(I) $R_i \leftrightarrow R_j$, $i \neq j$ (exchange 2 rows)	$-\det(A)$	sign switch
(II) $R_i \to \alpha R_i$, $\alpha \neq 0$ scalar (multiply a row by a nonzero scalar)	$\alpha \det(A)$	multiply by scalar α
(III) $R_i \to R_i + \beta R_j$, $i \neq j$ (Add to a row a scalar multiple of a different row)	$\det(A)$	No change

- Assuming we know $\det(A^T) = \det(A)$, we can derive similar rules for analogous column operations on $A \longleftrightarrow$ row operations on A^T.

TODAY:
1. $\det(AB) = \det A \det B$ (Product Rule)
2. Cramer's Rule for solving $Ax = b$ with A invertible via determinants

Product Rule

<u>Theorem 1</u>: A, B of size $n\times n$, then $\det(AB) = \det(A)\cdot\det(B)$.

<u>Example</u>: $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ $\det A = 2\cdot 3 - 2 = 4 \neq 0$

$B = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$ $\det B = (-1)(-2) - 3 = -1 \neq 0$

$\Big\}\ \det A \det B = -4$

$AB = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2+2 & 2\cdot 3-4 \\ -1+3 & 3-6 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -4 \end{bmatrix} \rightsquigarrow \det(AB) = -4$

- We saw a proof for general 2×2 matrices in Lecture 28.

<u>Consequence</u>: If A invertible then $AA^{-1} = I_n$, so $\det(A)\det(A^{-1}) = 1$.

Then $\det A \neq 0$ & $\det(A^{-1}) = \dfrac{1}{\det A}$. Moreover, if $\det(A) = 0$ then A is singular.

(we know this already)

Q: Why is the product formula valid in general?

- Special case: <u>AB singular matrix</u>. Then: A or B are singular (Lecture 9)

<u>Conclude</u> $\det(AB) = 0 = \det(A)\det(B)$. ($\det(\ ) = 0$ for the singular one).

Lemma: If A is singular of size $n\times n$, & B is any matrix of size $n\times n$, then AB is also singular.

Q: Why is this true? Use $\boxed{C \text{ singular if and only if } C^T \text{ is singular}}$

(Reason: If C invertible, then $(C^{-1})^T = (C^T)^{-1}$ so C^T is also invertible)

. Since A is singular, so is A^T. Now $(AB)^T = B^T A^T$

so $B^T A^T$ is singular (Pick $\vec{x} \neq \vec{0}$ in $\mathbb{R}^n$ with $A^T\vec{x} = \vec{0}$. Then $(B^T A^T)\vec{x} = B^T\underbrace{(A^T\vec{x})}_{=\vec{0}} = \vec{0}$)

. Then $AB = (B^T A^T)^T$ is singular

Proof of the Product Rule:

1. If A is singular, then AB is also singular, so $\det(AB) = 0 = \overbrace{\det(A)}^{=0}\det(B)$.
2. If A is not singular, then $A \sim I_n$ with a sequence of row operations.
$\rightsquigarrow \beta\det(A) = \det I_n = 1$ for some $\beta \neq 0$ (coming from rule rules)

(*) BUT The same row operations give $AB \sim B$ so $\beta \det(AB) = \det(B)$

since $\beta = \frac{1}{\det A}$, we get $\det(AB) = \det(B)/\beta = \det A \det B$.

. Let's check our claim (*) in an example.

<u>Example</u> $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$ $\rightsquigarrow$ $AB = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix}$

We write the operations needed to turn A into I_2 & check the same sequence of elementary row operations turns AB into B.

$$\underset{\det A}{A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}} \xrightarrow[R_2 \leftrightarrow R_1]{(\mathrm{I})} \underset{-\det A}{\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}} \xrightarrow[R_2 \to R_2 - 2R_1]{(\mathrm{III})} \underset{-\det A}{\begin{bmatrix} 1 & 3 \\ 0 & -4 \end{bmatrix}} \xrightarrow[R_2 \to \frac{R_2}{-4}]{(\mathrm{II})} \underset{\left(\frac{-1}{4}\right)(-\det A)}{\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}} \xrightarrow[R_1 \to R_1 - 3R_2]{(\mathrm{III})} \underset{\frac{1}{4}\det A}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2}$$

$$\underset{\det AB}{AB = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix}} \xrightarrow[R_2 \leftrightarrow R_1]{(\mathrm{I})} \underset{-\det AB}{\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}} \xrightarrow[R_2 \to R_2 - 2R_1]{(\mathrm{III})} \underset{-\det (AB)}{\begin{bmatrix} 2 & -3 \\ -4 & 8 \end{bmatrix}} \xrightarrow[R_2 \to \frac{R_2}{-4}]{(\mathrm{II})} \underset{\frac{-1}{4}(-\det AB)}{\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}} \xrightarrow[R_1 \to R_1 - 3R_2]{(\mathrm{III})} \underset{\frac{1}{4}\det AB}{\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} = B}$$

• <u>In general</u>: Need to see that $A \xrightarrow[op]{} A'$ gives $AB \xrightarrow[\text{same op}]{} A'B$

for any of the 3 elementary row operations.

After this fact, we concatenate all operations and get $A' = I_n$ so $A'B = B$.

Cramer's Rule

GOAL Given a nonsingular $n \times n$ matrix A, we want to use determinants to find the unique solution to $A\underline{x} = \underline{b}$ for a fixed $\underline{b}$ in $\mathbb{R}^n$.

Theorem 2 (Cramer's Rule) Write $A = [A_1 \cdots A_n]$ (non-singular) & $\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

For each $i = 1, 2, \ldots, n$ we build a new matrix

$$B_i = [A_1 \cdots A_{i-1} \; \underline{b}_i \; A_{i+1} \cdots A_n]$$ (replace i^{th} col of A with $\underline{b}$)

Then, the unique solution to $A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ satisfies:

$$x_1 = \frac{\det(B_1)}{\det(A)}, \quad x_2 = \frac{\det(B_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(B_n)}{\det(A)}.$$

Example: 1 Solve $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$B_1 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \rightsquigarrow \det(B_1) = 3 - 4 = -1$

$B_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightsquigarrow \det(B_2) = 4 - 1 = 3$

$\det A = 6 - 2 = 4 \neq 0$

$$\boxed{x_1 = \frac{-1}{4}, \quad x_2 = \frac{3}{4}}$$

Check: $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{4} + \frac{6}{4} \\ -\frac{1}{4} + \frac{9}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ✓

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Then, the unique solution to $A\begin{bmatrix} x_1\\ \vdots \\ x_n\end{bmatrix}=\begin{bmatrix} b_1\\ \vdots \\ b_n\end{bmatrix}$ satisfies:

$$x_1=\frac{\det(B_1)}{\det(A)},\quad x_2=\frac{\det(B_2)}{\det(A)},\ \ldots,\quad x_n=\frac{\det(B_n)}{\det(A)}.$$

Example: 2 Solve $\begin{bmatrix} 2&2\\1&3\end{bmatrix}\begin{bmatrix} x_1\\x_2\end{bmatrix}=\begin{bmatrix} 3\\4\end{bmatrix}$

$B_1=\begin{bmatrix} 3&2\\4&3\end{bmatrix} \rightsquigarrow \det(B_1)=9-8=1$

$B_2=\begin{bmatrix} 2&3\\1&4\end{bmatrix} \rightsquigarrow \det(B_2)=8-3=5$

$\det A = 6-2=4\neq 0$

$$\boxed{x_1=\frac{1}{4},\quad x_2=\frac{5}{4}}$$

Check: $\begin{bmatrix} 2&2\\1&3\end{bmatrix}\begin{bmatrix} 1/4\\5/4\end{bmatrix}=\begin{bmatrix} \frac{2}{4}+\frac{10}{4}\\ \frac{1}{4}+\frac{15}{4}\end{bmatrix}=\begin{bmatrix} 3\\4\end{bmatrix}$ ✓

Proof of Cramer's Rule

$A = [A_1 \cdots A_n]$ nonsingular

$A\underline{x} = \underline{b}$

- We check $x_1 = \dfrac{\det([\underline{b}\ A_2 \cdots A_n])}{\det(A)}$. Others are similar.

- $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \operatorname{col}_1 A + x_2 \operatorname{col}_2 A + \cdots + x_n \operatorname{col}_n A = x_1 A_1 + \cdots + x_n A_n.$

$$\underline{b} = \underbrace{x_1 A_1 + \cdots + x_n}_{\text{scalars!}} A_n.$$

So $\underline{b}^T = (x_1 A_1 + \cdots + x_n A_n)^T = x_1 A_1^T + \cdots + x_n A_n^T$ $1 \times n$ matrix

$\leadsto [b_1 \cdots b_n] = [x_1 \cdots x_n] \begin{bmatrix} A_1^T \\ \vdots \\ A_n^T \end{bmatrix} = [x_1 \cdots x_n]\, A^T.$

- Next, we build a new matrix C from A^T, by operation (II) $R_1 \to x_1 R_1$

$$A^T \xrightarrow[R_1 \to x_1 R_1]{} C = \begin{bmatrix} x_1 A_1^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix}$$

$\leadsto \det C = x_1 \det(A^T) = x_1 \det(A).$

But $x_1 A_1^T = \underline{b}^T - x_2 A_2^T - \cdots - x_n A_n^T$

So far: $\left\{ \begin{array}{l} x_1 A_1^T = \underline{b}^T - x_2 A_2^T - \cdots - x_n A_n^T \\ \det \begin{bmatrix} x_1 A_1^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix} = x_1 \det(A) \end{array} \right.$

$$\begin{bmatrix} x_1 A_1^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix} = \underset{x_1 \det(A)}{\begin{bmatrix} \underline{b}^T - x_2 A_2^T - \cdots - x_n A_n^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix}} \xrightarrow[R_1 \to R_1 - x_n R_n]{(\text{III})} \underset{x_1 \det(A)}{\begin{bmatrix} \underline{b}^T - x_2 A_2^T - \cdots - x_{n-1} A_{n-1}^T \\ A_2^T \\ \vdots \\ A_{n-1}^T \\ A_n^T \end{bmatrix}}$$

$$\xrightarrow[R_1 \to R_1 - x_{n-1} R_{n-1}^T]{(\text{III})} \underset{x_1 \det(A)}{\begin{bmatrix} \underline{b}^T - x_2 A_2^T - \cdots - x_{n-2} A_{n-2}^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix}} \xrightarrow{(\text{III})} \cdots \xrightarrow[R_1 \to R_1 - x_2 R_2]{(\text{III})} \underset{x_1 \det(A)}{\begin{bmatrix} \underline{b}^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix}} = B_1^T$$

So $\det B_1 = \det B_1^T = x_1 \det(A)$ gives $\boxed{x_1 = \dfrac{\det B_1}{\det A}}$!!