Lecture 31. §4.1-2 The eigenvalue problem for 2x2 matrices

Q: What is the eigenvalue (EV) problem?

- The name comes from German ("eigen" = "self")

<u>The EV Problem</u> Fix A $n\times n$ matrix. We want to find those lines in $\mathbb{R}^n$ through the origin $(= Sp(\vec{v})$ for $\vec{v} \neq \vec{0})$ that are invariant under the linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$. $T(\vec{x}) = A\vec{x}$

EV Problem v2: Find λ in $\mathbb{R}$ with $\mathcal{N}(A-\lambda I_n) \neq \{\vec{0}\}$.

Motivation

1. Solving differential equations
2. Analyzing population growth
3. Calculating powers of matrices: $A^2, A^3, \dots, A^{100}, \dots$
4. Simplify & draw conics in the plane (Lecture 5).

   conic $ax^2 + by^2 + cxy + dx + ey + f = 0$ (a,b,c,d,e,f fixed parameters)
5. Diagonalize linear transformations $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Q: What does "Diagonalize $T:\mathbb{R}^n \longrightarrow \mathbb{R}^n$" mean? $T(\vec{x}) = A\vec{x}$

<u>Conclusion:</u> To diagonalize a linear transformation $T:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ means finding a basis B for $\mathbb{R}^n$ consisting of eigenvectors ($[T]_{BB}$ diag!)

⚠ Not always possible!

Strategies for solving the EV Problem

EV Problem: Find λ with $\mathcal{N}(A-\lambda I_n) \neq \{\vec{0}\}$

so $A-\lambda I_n$ is a singular $n \times n$ matrix $\rightsquigarrow$ use det to check!

STRATEGY:

① Find λ with $\boxed{\det(A-\lambda I_n) = 0}$ (EIGENVALUES)

② For each value λ from ①, find $\boxed{E_\lambda = \mathcal{N}(A-\lambda I_n)}$

EIGENSPACE OF THE EIGENVALUE λ.

Any $\vec{v}$ in E_λ, $\vec{v} \neq \vec{0}$ will be an eigenvector

③ Collecting the bases of all E_λ will either

- allow us to diagonalize $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\vec{x} \longrightarrow A\vec{x}$

or

- show T is not diagonalizable.

The EV Problem for 2x2 matrices

- $\det(A-\lambda I_2)=0$
- Find $\mathcal{N}(A-\lambda I_2)$

Example 1

$$A=\begin{bmatrix}5 & -1\\ 8 & -1\end{bmatrix}$$

Example 2 $A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

$$\boxed{\text{An example for } 3\times 3 \text{ matrices}}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix}$$