Lecture 31: §4.1-2 The eigenvalue problem for 2x2 matrices

Q: What is the eigenvalue (EV) problem?

- The name comes from German ("eigen" = "self")

The EV Problem: Fix A $n\times n$ matrix. We want to find those lines in $\mathbb{R}^n$ through the origin ($= Sp(\vec{v})$ for $\vec{v} \neq \vec{0}$) that are invariant under the linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$. $T(\vec{x}) = A\vec{x}$

That is: find $\vec{v} \neq \vec{0}$ with $T(Sp(\vec{v}))$ lies in $Sp(\vec{v})$

This means: $A\cdot\vec{v}$ must be a scalar multiple of $\vec{v}$:

$$\boxed{\text{EV Problem v1: Find } \vec{v} \neq \vec{0} \text{ where } A\vec{v} = \lambda\vec{v} \text{ for some } \lambda \text{ in } \mathbb{R}}$$

Names: λ = eigenvalue, $\vec{v}$ = eigenvector ($\vec{0}$ always works, so we exclude it)

Equivalently: Find λ with $(A - \lambda I_n)\vec{v} = \vec{0}$ & $\vec{v} \neq \vec{0}$

$$\boxed{\text{EV Problem v2: Find } \lambda \text{ in } \mathbb{R} \text{ with } \mathcal{N}(A - \lambda I_n) \neq \{\vec{0}\}.}$$

EV Problem v2: Find λ in $\mathbb{R}$ with $\mathcal{N}(A-\lambda I_n) \neq \{\vec{0}\}$.

Example 1: $\lambda = 0$ eigenvalue means $\mathcal{N}(A - 0 I_n) = \mathcal{N}(A) \neq \{\vec{0}\}$.

Example 2: $A = I_n$ $\quad A - \lambda I_n = I_n - \lambda I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

so $A - \lambda I_n = \begin{pmatrix} 1-\lambda & & & 0 \\ & 1-\lambda & & \\ & & \ddots & \\ 0 & & & 1-\lambda \end{pmatrix}$ singular if & only if $\det(A-\lambda I_n) = 0$

But $\det(A - \lambda I_n) = \underbrace{(1-\lambda)(1-\lambda)\cdots(1-\lambda)}_{n \text{ times}} = (1-\lambda)^n \rightsquigarrow \lambda = 1$ is its single eigenvalue

- For $\lambda = 1$, $\mathcal{N}(A - \lambda I_n) = \mathcal{N}(\mathbb{O}) = \mathbb{R}^n$ (all vectors are eigenvectors)

Motivation

1. Solving differential equations
2. Analyzing population growth
3. Calculating powers of matrices: $A^2, A^3, \ldots, A^{100}, \ldots$
4. Simplify & draw conics in the plane (Lecture 5).
   conic $ax^2 + by^2 + cxy + dx + ey + f = 0$ (a, b, c, d, e, f fixed parameters)

§4.7 $\longrightarrow$ 5. Diagonalize linear transformations $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Q: What does " Diagonalize $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ " mean? $T(\vec{x}) = A\vec{x}$

A: Say we have a basis B for $\mathbb{R}^n$ consisting of eigenvectors $\vec{v_1}, \dots, \vec{v_n}$

$$\rightsquigarrow \begin{aligned} A\vec{v_1} &= \lambda_1 \vec{v_1} \\ A\vec{v_2} &= \lambda_2 \vec{v_2} \\ &\vdots \\ A\vec{v_n} &= \lambda_n \vec{v_n} \end{aligned} \rightsquigarrow [T]_{BB} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \lambda_n \end{bmatrix} \quad \text{vs } [T]_{EE} = A, \quad E = \{\vec{e_1}, \dots, \vec{e_n}\}$$

is a diagonal matrix.

Conclusion: To diagonalize a linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ means finding a basis B for $\mathbb{R}^n$ consisting of eigenvectors ($[T]_{BB}$ diag!)

⚠ Not always possible! Example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\vec{x} \longmapsto A\vec{x}$

- Only 1 eigenvalue: $\lambda = 1$ & $\dim \mathcal{N}(A - I_2) = 1$ $\rightsquigarrow$ no basis of eigenvectors!
- $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightsquigarrow \begin{cases} x_1 + x_2 = \lambda x_1 & (1) \\ x_2 = \lambda x_2 & (2) \end{cases}$

(2) $\lambda = 1$ $\rightsquigarrow$ If $\lambda = 1$: (1) gives $x_2 = 0$ $\mathcal{N}(A - I_2) = Sp\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

or $x_2 = 0$ $\rightsquigarrow$ (1) gives $x_1 = 0$ or $\lambda = 1$

Strategies for solving the EV Problem

EV Problem: Find λ with $\mathcal{N}(A-\lambda I_n) \neq \{\vec{0}\}$.

so $A-\lambda I_n$ is a singular $n \times n$ matrix $\rightsquigarrow$ use det to check!

STRATEGY:

① Find λ with $\boxed{\det(A-\lambda I_n)=0}$ (EIGENVALUES)

② For each value λ from ①, find $\boxed{E_\lambda = \mathcal{N}(A-\lambda I_n)}$

EIGENSPACE OF THE EIGENVALUE λ.

Any $\vec{v}$ in E_λ, $\vec{v} \neq \vec{0}$ will be an eigenvector

③ Collecting the bases of all E_λ will either

- allow us to diagonalize $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\vec{x} \longmapsto A\vec{x}$

or

- show T is not diagonalizable.

The EV Problem for 2x2 matrices

- $\det(A-\lambda I_2)=0$
- Find $\mathcal{N}(A-\lambda I_2)$

Example ① $A=\begin{bmatrix}5 & -1\\ 8 & -1\end{bmatrix}$

Diagonalizable : $B=\left\{\begin{bmatrix}1\\4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\right\}$ basis for $\mathbb{R}^2$.

- $\det(A-\lambda I_2)=\det\left(\begin{bmatrix}5 & -1\\ 8 & -1\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)=\det\left(\begin{bmatrix}5-\lambda & -1\\ 8 & -1-\lambda\end{bmatrix}\right)$

$=(5-\lambda)(-1-\lambda)-8(-1)=(-5)-5\lambda+\lambda+\lambda^2+8=\lambda^2-4\lambda+3$

Solns: $\lambda=\dfrac{-(-4)\pm\sqrt{(-4)^2-4\cdot1\cdot3}}{2\cdot1}=\dfrac{4\pm\sqrt{16-12}}{2}=\dfrac{4\pm2}{2}\begin{smallmatrix}\nearrow 3\\ \searrow 1\end{smallmatrix}$

$\rightsquigarrow$ 2 eigenvalues : $\lambda=3$ & $\lambda=1$.

- $E_1=\mathcal{N}(A-1I_2)=\mathcal{N}\left(\begin{bmatrix}5-1 & -1\\ 8 & -1-1\end{bmatrix}\right)=\mathcal{N}\begin{pmatrix}4 & -1\\ 8 & -2\end{pmatrix}$

$\left[\begin{array}{cc|c}4 & -1 & 0\\ 8 & -2 & 0\end{array}\right]\xrightarrow[R_2\to R_2-2R_1]{}\left[\begin{array}{cc|c}4 & -1 & 0\\ 0 & 0 & 0\end{array}\right]\xrightarrow[R_1\to \frac{R_1}{4}]{}\left[\begin{array}{cc|c}1 & -1/4 & 0\\ 0 & 0 & 0\end{array}\right]$ REF

x_1 dep, x_2 indep

$x_1=\frac{1}{4}x_2 \rightsquigarrow \begin{bmatrix}x_1\\x_2\end{bmatrix}=x_2\begin{bmatrix}1/4\\1\end{bmatrix}$

$\mathcal{N}(A-I_2)=Sp\left(\begin{bmatrix}1/4\\1\end{bmatrix}\right)=Sp\left(\begin{bmatrix}1\\4\end{bmatrix}\right)$

- $E_3=\mathcal{N}(A-3I_2)=\mathcal{N}\begin{pmatrix}2 & -1\\ 8 & -4\end{pmatrix}$ $\quad\begin{bmatrix}2 & -1\\ 8 & -4\end{bmatrix}\xrightarrow[R_2\to R_2-4R_1]{}\begin{bmatrix}2 & -1\\ 0 & 0\end{bmatrix}$ $\quad 2x_1=x_2$

$=Sp\left(\begin{bmatrix}1\\2\end{bmatrix}\right).$

Example 2 $\quad A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

Guess $\lambda = 3$ & -1 are eigenvalues & $\begin{cases} E_3 = Sp\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \\ E_{-1} = Sp\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\end{cases}$ eigenspaces

- $\det\left(A - \lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}\right) = \det\begin{pmatrix}3-\lambda & 0\\ 0 & -1-\lambda\end{pmatrix} = (3-\lambda)(-1-\lambda) = 0$

So only solutions : $\lambda = 3$, $\lambda = -1$.

- $E_3 = \mathcal{N}(A - 3I_2) = \mathcal{N}\left(\begin{bmatrix}0&0\\0&-1\end{bmatrix}\right) = Sp(e_1) \quad \begin{pmatrix} x_1 \text{ indep.} \\ \underset{=0}{x_2} \text{ dep} \end{pmatrix}$
- $E_{-1} = \mathcal{N}(A - (-1)I_2) = \mathcal{N}\left(\begin{bmatrix}4&0\\0&0\end{bmatrix}\right) = Sp(e_2) \quad \begin{pmatrix} \overset{=0}{x_1} \text{ dep} \\ x_2 \text{ indep} \end{pmatrix}$

Obs: If $A = \begin{bmatrix} a & b \\ c & d\end{bmatrix} \rightsquigarrow$ Q What does $\det(A - \lambda I_2)$ look like?

A: $\det(A - \lambda I_2) = \det\left(\begin{bmatrix}a&b\\c&d\end{bmatrix} - \begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}\right) = \det\begin{pmatrix}a-\lambda & b\\ c & d-\lambda\end{pmatrix} = (a-\lambda)(d-\lambda) - bc$

$$= ad - a\lambda - d\lambda + \lambda^2 - bc = \lambda^2 + \underbrace{(-a-d)}_{=p}\lambda + \underbrace{(ad-bc)}_{=q}$$

Polynomial of degree 2 in λ

$\rightsquigarrow$ **Solutions** : $\boxed{\lambda = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}}$

$\rightsquigarrow p^2 - 4q > 0$: 2 solns

$\rightsquigarrow p^2 - 4q = 0$ 1 (double) solution

$\rightsquigarrow p^2 - 4q < 0$ complex soln (§4.6)

An example for 3x3 matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \rightsquigarrow \text{Want to solve } \det(A - \lambda I_3) = 0$$

$$A - \lambda I_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & -2 \\ 0 & 1 & -2-\lambda \end{bmatrix}$$

$$\rightsquigarrow \det(A-\lambda I_3) = (1-\lambda)\det\left(\begin{bmatrix} 1-\lambda & -2 \\ 1 & -2-\lambda \end{bmatrix}\right) - 1 \underbrace{\det\left(\begin{bmatrix} 0 & -2 \\ 0 & -2-\lambda \end{bmatrix}\right)}_{=0}$$

$$= (1-\lambda)\left((1-\lambda)(-2-\lambda)+2\right)$$

$$= (1-\lambda)\left(\lambda^2 + (2-1)\lambda - 2 + 2\right) = (1-\lambda)\lambda(\lambda+1)$$

$\rightsquigarrow$ 3 eigenvalues: $\lambda = 1$, $\lambda = 0$, $\lambda = -1$.

- $E_0 = \mathcal{N}(A) = \operatorname{Sp}\left(\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}\right)$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_2]{} \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{REF}}$$

x_1, x_2 dep
x_3 indep

$x_1 = -2x_3$
$x_2 = 2x_3$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \qquad \lambda = 0, 1, -1 \quad \text{eigenvalues}$$

$$\bullet\ E_1 = \mathcal{N}(A - I_3) = \mathcal{N}\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}\right) = Sp\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_3]{} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[\substack{R_3 \to -R_3 \\ R_2 \leftrightarrow R_3}]{} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{REF}$$

x_2, x_3 dep
x_1 indep

Alternative: rank $(A - I_3) = 2$ so nullity $= 1$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $\mathcal{N}(A - I_3)$, so it must be a basis for E_1.

$$\bullet\ E_{-1} = \mathcal{N}(A + I_3) = \mathcal{N}\left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{bmatrix}\right) = Sp\left(\begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_3]{} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_3]{} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[\substack{R_2 \leftrightarrow R_3 \\ R_1 \to R_1/2}]{} \underbrace{\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{REF}$$

x_1, x_2 dep , x_3 indep

$$\mathcal{B} = \left\{ \underset{(E_0)}{\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}}, \underset{(E_1)}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{(E_{-1})}{\begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}} \right\}$$

is a basis of eigenvectors $\det\begin{pmatrix} 1 & -2 & -1/2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 1$

so $[T]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\quad T(\vec{x}) = A\vec{x}$.