Lecture 32: §4.4 Eigenvalues and the characteristic polynomial

Last Time: Stated the Eigenvalue problem in 2 ways & did examples.

$$\boxed{\text{EV Problem v1: Find } \vec{v} \neq \vec{0} \text{ where } A\vec{v} = \lambda \vec{v} \text{ for some } \lambda \text{ in } \mathbb{R}}$$

Names: λ = eigenvalue, $\vec{v}$ = eigenvector ($\vec{0}$ always works, so we exclude it)

$$\boxed{\text{EV Problem v2: Find } \lambda \text{ in } \mathbb{R} \text{ with } \mathcal{N}(A - \lambda I_n) \neq \{\vec{0}\}.}$$

so $A - \lambda I_n$ is a singular $n \times n$ matrix

STRATEGY:

1. Find λ with $\boxed{\det(A - \lambda I_n) = 0}$ (EIGENVALUES)

2. For each value λ from 1, find $\boxed{E_\lambda = \mathcal{N}(A - \lambda I_n)}$

   EIGENSPACE OF THE EIGENVALUE λ.

Any $\vec{v}$ in E_λ, $\vec{v} \neq \vec{0}$ will be an eigenvector

EXAMPLE:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \qquad \det(A) = 0$$

The Characteristic Polynomial

$A = n \times n$ matrix

Def $P_A(\lambda) = \det(A - \lambda I_n) =$ characteristic polynomial of A.

- $P_A(\lambda)$ is a polynomial in λ (with real coefficients).

Q: What else can we say about $P_A(\lambda)$?

Theorem 1: The eigenvalues of A are the zeroes (or roots) of $P_A(\lambda)$

Theorem 2: $P_A(\lambda)$ is a polynomial of degree n, coeff $X^n = (-1)^n$, $P_A(0) = \det(A)$.

$$P_A(\lambda) = \det(A - \lambda I_n) \quad \text{poly in } \lambda \text{ of degree } n$$
$$= (-1)^n \lambda^n + b_{n-1} \lambda^{n-1} + \cdots + b_1 \lambda + (\det A).$$

Q1: How many real roots does $P_A(\lambda)$ have?

Q2: How can we find them?

Properties of Eigenvalues

Q: • How do eigenvalues of A relate to eigenvalues of $A^2, A^3, A^4, \ldots$?

• ____ A^{-1}?

Theorem 3: Fix an $n \times n$ matrix A & let λ be an eigenvalue of A.

Then: ① λ^k is an eigenvalue of A^k for $k = 2, 3, 4, \ldots$

② If A is nonsingular, then $\lambda \neq 0$ & $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}!

Moreover, they all share at least one eigenvector.

Theorem 4: A $n \times n$ matrix with λ eigenvalue. Then, for any scalar μ the scalar $(\lambda+\mu)$ is an eigenvalue for $A+\mu I_n$.
Furthermore, both matrices share the corresponding eigenvectors.

EXAMPLE $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ $\quad P_A(\lambda) = -\lambda(\lambda-2)(\lambda-3)$