Lecture 32: §4.4 Eigenvalues and the characteristic polynomial

Last Time: Stated the Eigenvalue problem in 2 ways & did examples.

$$\boxed{\text{EV Problem v1: Find } \vec{v} \neq \vec{0} \text{ where } A\vec{v} = \lambda\vec{v} \text{ for some } \lambda \text{ in } \mathbb{R}}$$

Names: λ = eigenvalue , $\vec{v}$ = eigenvector ($\vec{0}$ always works, so we exclude it)

$$\boxed{\text{EV Problem v2: Find } \lambda \text{ in } \mathbb{R} \text{ with } \mathcal{N}(A-\lambda I_n) \neq \{\vec{0}\}.}$$

so $A-\lambda I_n$ is a singular $n\times n$ matrix $\rightsquigarrow$ use det to check!

STRATEGY:

CHARACTERISTIC POLYNOMIAL (in λ)

① Find λ with $\boxed{\det(A-\lambda I_n)=0}$ (EIGENVALUES)

② For each value λ from ①, find $\boxed{E_\lambda = \mathcal{N}(A-\lambda I_n)}$

EIGENSPACE OF THE EIGENVALUE λ.

Any $\vec{v}$ in E_λ, $\vec{v} \neq \vec{0}$ will be an eigenvector

EXAMPLE: $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ $\det(A) = 0$ ($\rightsquigarrow$ A is singular so $\lambda = 0$ is an eigenvalue!)

$$A - \lambda I_3 = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & -2 \\ 1 & 3-\lambda & 1 \\ 1 & 3 & 1-\lambda \end{bmatrix}$$

Use cofactor exp.

- $\det(A - \lambda I_3) = (1-\lambda) \det \begin{bmatrix} 3-\lambda & 1 \\ 3 & 1-\lambda \end{bmatrix} + (-2) \det \begin{bmatrix} 1 & 3-\lambda \\ 1 & 3 \end{bmatrix}$

$$= (1-\lambda)\left((1-\lambda)(3-\lambda) - 3\right) - 2\left(3 - (3-\lambda)\right)$$

$$= (1-\lambda)(\lambda^2 - 4\lambda) - 2\lambda = \lambda\left((1-\lambda)(\lambda-4) - 2\right)$$

$$= \lambda(-\lambda^2 + 5\lambda - 4 - 2) = \lambda(-\lambda^2 + 5\lambda - 6) = -\lambda(\lambda-2)(\lambda-3)$$

$\rightsquigarrow$ Eigenvalues: $\lambda = 0, 2, 3$.

Poly in λ of degree 3.

- $E_0 = \mathcal{N}(A - 0 I_3) = \mathrm{Sp}\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right)$

$$\begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 \to R_2/3]{} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 2x_3$ x_3 free
$x_2 = -x_3$

$E_2 = \mathcal{N}(A - 2I_3) = \mathrm{Sp}\left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right)$ & $E_3 = \mathcal{N}(A - 3I_3) = \mathrm{Sp}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right)$

- $B = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$ and $[T]_{BB} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $T : \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{x} \mapsto A\vec{x}$.

The Characteristic Polynomial

$A = n\times n$ matrix

Def $P_A(\lambda) = \det(A - \lambda I_n) =$ characteristic polynomial of A.

- $P_A(\lambda)$ is a polynomial in λ (with real coefficients).

Ex $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow P_A(\lambda) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \lambda^2 + \underbrace{(-a-d)}_{=-\mathrm{tr}(A)}\lambda + \underbrace{(ad-bc)}_{=\det A}$

Q: What else can we say about $P_A(\lambda)$?

Theorem 1: The eigenvalues of A are the zeroes (or roots) of $P_A(\lambda)$

Theorem 2: $P_A(\lambda)$ is a polynomial of degree n, coeff $X^n = (-1)^n$, $P_A(0) = \det(A)$

Why? • $P_A(0) = \det(A - 0I_n) = \det(A)$ ✓

$P_{M(1,1)} = (-1)^{n-1}\lambda^{n-1} + (\text{lot})$

• $P_A(\lambda) = \det\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & & a_{1n} \\ a_{21} & a_{22}-\lambda & & & \\ \vdots & & & & \\ a_{n1} & & \cdots & a_{n(n-1)} & a_{nn}-\lambda \end{bmatrix} = (a_{11}-\lambda)\det\begin{bmatrix} a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$

$-a_{12}\det\begin{bmatrix} a_{21} & a_{23} & & \\ a_{31} & a_{33}-\lambda & \ddots & \\ \vdots & & & \\ a_{n1} & & & a_{nn}-\lambda \end{bmatrix} + \cdots + (-1)^{1+n} a_{1n}\det\begin{bmatrix} a_{21} & a_{22}-\lambda & \cdots & a_{2n-1} \\ & & & \vdots \\ & & & a_{n-1,n-1}-\lambda \\ a_{n1} & \cdots & \cdots & a_{n(n-1)} \end{bmatrix} \rightsquigarrow \boxed{\lambda^n \text{ term with } (-1)^n.}$

$\rightarrow$ no λ^n term $\leftarrow$

$$P_A(\lambda) = \det(A - \lambda I_n) \quad \text{poly in } \lambda \text{ of degree } n$$
$$= (-1)^n \lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_1 x + (\det A).$$

Q1: How many real roots does $P_A(\lambda)$ have?

A: At most n ($=$ degree of P_A).

Q2: How can we find them?

A: No complete list of rules/methods, but we have some heuristics

EXAMPLE 1 $A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$

$$P_A(\lambda) = \det\left(\begin{bmatrix} -2-\lambda & -1 \\ 1 & -2-\lambda \end{bmatrix}\right) = (-2-\lambda)(-2-\lambda) + 1 = \lambda^2 + 4\lambda + 5$$

- Quadratic formula: $\dfrac{-4 \pm \sqrt{16-20}}{2} = \dfrac{-4 \pm \sqrt{\overset{<0}{-4}}}{2}$ no real roots!

- Another way to see this: $P_A(\lambda) = \underbrace{(-\lambda-2)^2}_{\geq 0} + \underbrace{1}_{>0} > 0$ always!

(it's a sum of squares!)

EXAMPLE 2

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$P_A(\lambda) = \det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -2-\lambda & -1 \\ 0 & 1 & -2-\lambda \end{bmatrix}\right) = (1-\lambda)\det\left(\begin{bmatrix} -2-\lambda & -1 \\ 1 & -2-\lambda \end{bmatrix}\right) = \underbrace{(1-\lambda)}_{\lambda=1 \text{ real root}}\underbrace{(\lambda^2-4\lambda+5)}_{\text{NO real roots}}$$

"block decomposition"

EXAMPLE 3

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

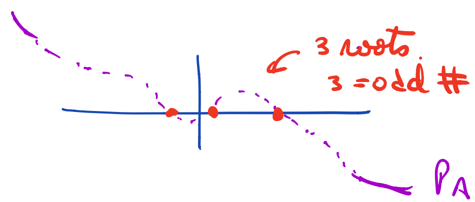
$$P_A(\lambda) = \det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{bmatrix}\right) = (1-\lambda)(2-\lambda)(-3-\lambda)$$

In general: If n is ODD, then $P_A(\lambda)$ has at least 1 real root ("odd many" of them)

Why? $P_A(\lambda) = \underbrace{(-1)^n}_{=-1}\lambda^n + (\text{lower order terms})$

- $\lim_{\lambda\to\infty} P_A(\lambda) = -\infty$ & $\lim_{\lambda\to-\infty} P_A(\lambda) = -(-\infty) = \infty$
- P_A is continuous

so "in between" we have to cross the x-axis (odd # of times) (Intermediate Value Thm)



Properties of Eigenvalues

Q: • How do eigenvalues of A relate to eigenvalues of $A^2, A^3, A^4, \ldots$?
• ______________________________ A^{-1}?

Theorem 3: Fix an $n\times n$ matrix A & let λ be an eigenvalue of A.
Then: ① λ^k is an eigenvalue of A^k for $k = 2, 3, 4, \ldots$
② If A is nonsingular, then $\lambda \neq 0$ & $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.
Moreover, they all share at least one eigenvector.

Why? Pick $\vec{v} \neq \vec{0}$ in $E_\lambda(A)$. Then: $A\vec{v} = \lambda\vec{v}$

① $k=2$ $A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda A\vec{v} = \lambda\lambda\vec{v} = \lambda^2\vec{v}$.
So $\vec{v} \in E_{\lambda^2}(A^2)$. Iterating this gives $A^3\vec{v} = \lambda^3\vec{v}$, etc.

② Since A is invertible, $\mathcal{N}(A) = \{\vec{0}\}$ & $P_A(0) = \det A \neq 0$.
so $\lambda \neq 0$.
Then: $\vec{v} = A^{-1}\lambda\vec{v} = \lambda A^{-1}\vec{v}$ so $\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$ so $\vec{v}$ in $E_{\frac{1}{\lambda}}(A^{-1})$.

Theorem 4: A $n\times n$ matrix with λ eigenvalue. Then, for any scalar μ the scalar $(\lambda+\mu)$ is an eigenvalue for $A+\mu I_n$.
Furthermore, both matrices share the corresponding eigenvectors.

Why? Pick $v \neq \vec{0}$ with

$$\begin{array}{rl} A\vec{v} &= \lambda\vec{v} \\ + \quad \mu\vec{v} &= \mu\vec{v} \\ \hline \end{array}$$

$$(A+\mu I_n)\vec{v} = A\vec{v}+\mu\vec{v} = (\lambda+\mu)\,\vec{v}$$

so $\vec{v}$ in $E_\lambda(A)$ gives $\vec{v}$ in $E_{\lambda+\mu}(A+\mu I_n)$

Now $\vec{w}$ in $E_{\lambda+\mu}(\underbrace{A+\mu I_n}_{A_{new}})$ gives $\vec{w}$ in $E_{\lambda+\mu+(-\mu)}(A+\mu I_n-\mu I_n)$
$= E_{\lambda+\mu}(A)$

EXAMPLE $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ $\quad P_A(\lambda) = -\lambda(\lambda-2)(\lambda-3)$

- A singular ($\lambda=0$ eigenvalue)
- $0^2, 2^2, 3^2$ are eigenvalues of A^2 ; $0, 2^3, 3^3$ are eigenvalues of A^3, ...
- Eigenvalues of $A-5I_3$ include $0-5$, $2-5$, $3-5$, ie $-5, -3, -2$.
  $n=3$, so these are the ONLY eigenvalues of $A-5I_3$.