Lecture 33: §4.5 Eigenvalues and eigenvectors

Recall: A $n\times n$ matrix $\rightsquigarrow$ $P_A(\lambda) = \det(A-\lambda I_n)$ characteristic polynomial of A

- $P_A(\lambda)$ has degree n, $\text{coeff}(\lambda^n) = (\pm 1)^n$, $P_A(0) = \det(A)$.
- Eigenvalues of A (λ with $\mathcal{N}(A-\lambda I_n) \neq \{0\}$) $=$ Zeroes of $P_A(\lambda)$

Theorem 1: If λ is an eigenvalue of A, then:

① λ^2 is an eigenvalue of A^2, λ^3 ______ λ^3, etc.
So λ^k ______ A^k for any $k = 2, 3, 4, \ldots$

Moreover, $E_\lambda(A) \subseteq E_{\lambda^k}(A^k)$

② If A is nonsingular (invertible), then $\lambda \neq 0$ & $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.
Moreover, $E_\lambda(A) = E_{\lambda^{-1}}(A^{-1})$

③ For any μ scalar, $(\lambda+\mu)$ is an eigenvalue of the matrix $A+\mu I_n$
Moreover $E_\lambda(A) = E_{\lambda+\mu}(A+\mu I_n)$.

Theorem 2: $P_A(\lambda) = P_{A^T}(\lambda)$ as polynomials in λ.

In particular, A & A^T have the same eigenvalues, but typically different eigenspaces

EXAMPLE: $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ $\qquad A^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix}$

$$P_{A^T}(\lambda) = \det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 3-\lambda & 3 \\ -2 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda)\det\begin{pmatrix} 3-\lambda & 3 \\ 1 & 1-\lambda \end{pmatrix} - 1\det\begin{pmatrix} 0 & 3 \\ -2 & 1-\lambda \end{pmatrix} + \det\begin{pmatrix} 0 & 3-\lambda \\ -2 & 1 \end{pmatrix}$$

$$= (1-\lambda)\left((1-\lambda)(3-\lambda) - 3\right) - 6 + 2(3-\lambda) = (1-\lambda)(\lambda^2 - 4\lambda) - 2\lambda$$

$$= \lambda\left((1-\lambda)(\lambda-4) - 2\right) = -\lambda(\lambda-2)(\lambda-3) = P_A(\lambda)$$

$E_0(A) = \mathrm{Sp}\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right)$ $\qquad$ $E_0(A^T) = \mathrm{Sp}\left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right)$

$E_2(A) = \mathrm{Sp}\left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right)$ $\qquad$ vs, $\qquad$ $E_2(A^T) = \mathrm{Sp}\left(\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right)$

$E_3(A) = \mathrm{Sp}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right)$ $\qquad$ $E_3(A^T) = \mathrm{Sp}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)$

Very different eigenspaces!

$\boxed{\text{Eigenvectors}}$

<u>Recall</u>: $\vec{v} \neq \vec{0}$ is an eigenvector if $A\vec{v} = \lambda \vec{v}$ for some λ (eigenvalue)

$$E_\lambda(A) = \text{Span}\left(\{ \vec{v} \text{ with } A\vec{v} = \lambda\vec{v} \} \right) = \mathcal{N}(A - \lambda I_n).$$

This is how we compute E_λ.

EXAMPLE 1 $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

- <u>Step 1</u>: Compute eigenvalues:

- <u>Step 2</u>: For each eigenvalue λ, compute $E_\lambda(A)$

EXAMPLE 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

EXAMPLE 3:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

Eigenspaces

A $n \times n$ matrix

Def: Given λ scalar, set $E_\lambda = E_\lambda(A) = \mathcal{N}(A - \lambda I_n)$

Obs: If λ is an eigenvalue: $E_\lambda(A) \neq \{0\}$.

If λ is not an eigenvalue: $E_\lambda(A) = \{0\}$

Q: What is $\dim E_\lambda$ when λ is an eigenvalue?

Names: $\dim E_\lambda$ = the geometric multiplicity of λ,

$\text{mult}(\lambda, P_A)$ = the algebraic ________________ .

Earlier examples:

#	Matrix A	P_A	Eigenvalues	$\dim E_\lambda$	$\text{mult}(\lambda, P_A)$
1	$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$	$(\lambda+1)^2$			
2	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$	$(\lambda-2)(\lambda+1)^2$			
3	$\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 8 \\ 0 & 0 & 4 \end{bmatrix}$	$-(\lambda-2)(\lambda+1)(\lambda-4)$			

Linear independence for eigenvectors

Theorem: Fix A $n\times n$ matrix an a list of k distinct eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$. Pick one eigenvector $\vec{v}_j \neq \vec{0}$ for each λ_j. $(A\vec{v}_j = \lambda_j \vec{v}_j)$

Then: $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ is l.i. in $\mathbb{R}^n$.

Defective Matrices

- Fix A $n \times n$ matrix. We are interested in situations where:

$$(*) \quad \boxed{\text{alg mult}(\lambda) = \text{geom mult}(\lambda) \quad \text{for all } \lambda \text{ eigenvalues of } A}$$

<u>Def</u>: We say A is a <u>defective matrix</u> when $(*)$ fails. More precisely, A has a (real or complex) eigenvalue with alg mult$(\lambda) >$ geom mult(λ)

<u>Theorem</u>: alg mult$(\lambda) \geq$ geom mult(λ) for any eigenvalue of A
(HARD!)

<u>Consequence</u>: If all alg multiplicities are 1, then A is NOT defective.