Lecture 33: §4.5 Eigenvalues and eigenvectors

Recall: A $n\times n$ matrix $\rightsquigarrow$ $P_A(\lambda) = \det(A-\lambda I_n)$ characteristic polynomial of A

- $P_A(\lambda)$ has degree n, $\mathrm{coeff}(\lambda^n) = (\pm 1)^n$, $P_A(0) = \det(A)$.
- Eigenvalues of A (λ with $\mathcal{N}(A-\lambda I_n) \neq \{\mathbf{0}\}$) $=$ Zeroes of $P_A(\lambda)$

Theorem 1: If λ is an eigenvalue of A, then:

① λ^2 is an eigenvalue of A^2, λ^3 ______ A^3, etc.
So λ^k ______ A^k for any $k = 2, 3, 4, \ldots$

Moreover, $E_\lambda(A) \subseteq E_{\lambda^k}(A^k)$ (if $v \neq \vec{0}$ & $A\vec{v} = \lambda\vec{v}$, then $A^k\vec{v} = \lambda^k\vec{v}$)

② If A is nonsingular (invertible), then $\lambda \neq 0$ & $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.
Moreover, $E_\lambda(A) = E_{\lambda^{-1}}(A^{-1})$ (same eigenvectors)

③ For any μ scalar, $(\lambda+\mu)$ is an eigenvalue of the matrix $A+\mu I_n$
Moreover $E_\lambda(A) = E_{\lambda+\mu}(A+\mu I_n)$. (same eigenvectors)

Theorem 2: $P_A(\lambda) = P_{A^T}(\lambda)$ as polynomials in λ.

In particular, A & A^T have the same eigenvalues, but typically different eigenspaces

Why? Know $\det(C) = \det(C^T)$ for any $n\times n$ matrix

Take $C = A - \lambda I_n \rightsquigarrow C^T = (A-\lambda I_n)^T = A^T - \lambda I_n^T = A^T - \lambda I_n$

so $P_{A^T}(\lambda) = \det(A^T - \lambda I_n) = \det(A - \lambda I_n) = P_A(\lambda)$.

EXAMPLE: $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ $\qquad A^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix}$

$$P_{A^T}(\lambda) = \det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 3-\lambda & 3 \\ -2 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda)\det\begin{pmatrix} 3-\lambda & 3 \\ 1 & 1-\lambda \end{pmatrix} - 1\det\begin{pmatrix} 0 & 3 \\ -2 & 1-\lambda \end{pmatrix} + \det\begin{pmatrix} 0 & 3-\lambda \\ -2 & 1 \end{pmatrix}$$

$$= (1-\lambda)\left((1-\lambda)(3-\lambda) - 3\right) - 6 + 2(3-\lambda) = (1-\lambda)(\lambda^2 - 4\lambda) - 2\lambda$$

$$= \lambda\left((1-\lambda)(\lambda-4) - 2\right) = -\lambda(\lambda-2)(\lambda-3) = P_A(\lambda)$$

$E_0(A) = Sp\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right)$ $\qquad$ $E_0(A^T) = Sp\left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right)$

$E_2(A) = Sp\left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right)$ $\qquad$ vs $\qquad$ $E_2(A^T) = Sp\left(\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right)$

$E_3(A) = Sp\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right)$ $\qquad$ $E_3(A^T) = Sp\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)$

Very different eigenspaces!

Eigenvectors

<u>Recall</u>: $\vec{v} \neq \vec{0}$ is an eigenvector if $A\vec{v} = \lambda\vec{v}$ for some λ (eigenvalue)

$$E_\lambda(A) = \mathrm{Span}(\{\vec{v} \text{ with } A\vec{v} = \lambda\vec{v}\}) = \mathcal{N}(A - \lambda I_n).$$

This is how we compute E_λ.

<u>EXAMPLE</u> 1 $\quad A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

- <u>Step 1</u>: Compute eigenvalues:

$$P_A(\lambda) = \det\left(\begin{bmatrix} -1-\lambda & 1 \\ 0 & -1-\lambda \end{bmatrix}\right) = (-1-\lambda)^2 = (\lambda+1)^2 \rightsquigarrow \lambda = -1 \text{ (double root).}$$

- <u>Step 2</u>: For each eigenvalue λ, compute $E_\lambda(A)$

$$E_{-1}(A) = \mathcal{N}(A - (-1)I_n) = \mathcal{N}\left(\begin{bmatrix} 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}\right) = \mathrm{Sp}(e_1).$$

REF $\quad x_2 = 0.$

- So $\dim E_{-1}(A) = 1 < 2 = \text{mult of } (-1) \text{ as a root of } P_A.$

in general we have $\leq$ here.

- $\mathbb{R}^2$ does not have a basis of eigenvectors of A.

EXAMPLE 2 $\quad A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$$P_A(\lambda) = \det(A - \lambda I_3) = \det\left(\left[\begin{array}{c|cc} 2-\lambda & 0 & 0 \\ \hline 0 & -1-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{array}\right]\right) = (2-\lambda)^{1}(-1-\lambda)^{2}$$

"block decomposition"

$\rightsquigarrow$ 2 eigenvalues: $\lambda = 2$, $\lambda = -1$

multiplicity $=$ 1 , 2

- $E_{-1}(A) = \mathcal{N}(A - (-1)I_3) = \mathcal{N}\left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = Sp(e_2)$

REF $x_1 = x_3 = 0$

- $E_2(A) = \mathcal{N}(A - 2I_3) = \mathcal{N}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}\right) = Sp(e_1).$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3/-3 \\ R_2 \to R_2/-3}]{} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 + \frac{1}{3}R_3]{} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

REF $\qquad x_2 = x_3 = 0$

Summary:

- 2 eigenvalues $\quad mult(-1, P_A) = 2$, $mult(2, P_A) = 1$
- $\dim E_{-1} = 1$, $\dim E_2 = 1$
- $\dim E_{-1} < mult(-1, P_A)$ & $\dim E_2 = mult(2, P_A)$.
- $\mathbb{R}^3$ does not have a bases of eigenvectors of A.

EXAMPLE 3: $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 8 \\ 0 & 0 & 4 \end{bmatrix}$

• $P_A(\lambda) = \det(A - \lambda I_n) = \det \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & -1-\lambda & 8 \\ 0 & 0 & 4-\lambda \end{bmatrix} = (2-\lambda)(-1-\lambda)(4-\lambda)$

upper triangular.

We have 3 simple (= mult 1) roots:

3 eigenvalues: $\lambda = 2, -1, 4$.

• $E_{-1} = \mathcal{N}(A-(-1)I_3) = \mathcal{N}\left(\begin{bmatrix} 2+1 & 0 & 1 \\ 0 & 0 & 8 \\ 0 & 0 & 4+1 \end{bmatrix}\right) = Sp(e_2)$ (Eqn 2 $8x_3 = 0$, Eqn 1 : $3x_1 = 0$) x_2 free

• $E_2 = \mathcal{N}(A - 2I_3) = \mathcal{N}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & -3 & 8 \\ 0 & 0 & 6 \end{bmatrix}\right) = Sp(e_1)$ (Eqn 3 $6x_3 = 0$, Eqn 2 : $-3x_2 = 0$.) x_1 free

• $E_4 = \mathcal{N}(A - 4I_3) = \mathcal{N}\left(\begin{bmatrix} -2 & 0 & 1 \\ 0 & -5 & 8 \\ 0 & 0 & 0 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -8/5 \\ 0 & 0 & 0 \end{bmatrix}\right) = Sp\left(\begin{bmatrix} 5 \\ 16 \\ 10 \end{bmatrix}\right)$

$x_1 = +\frac{1}{2} x_3$, $x_2 = \frac{8}{5} x_3$

$B = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 16 \\ 10 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$ consisting of 3 eigenvectors of A:

Summary: • A has 3 distinct eigenvalues, all of multiplicity 1.
• $\dim E_{-1} = \dim E_2 = \dim E_4 = 1$
• Basis for $\mathbb{R}^3$ of eigenvectors of A.

Eigenspaces

A $n \times n$ matrix

Def: Given λ scalar, set $E_\lambda = E_\lambda(A) = \mathcal{N}(A - \lambda I_n)$ (subspace of $\mathbb{R}^n$)

Obs: If λ is an eigenvalue: $E_\lambda(A) \neq \{0\}$. ($\dim E_\lambda > 0$)

If λ is not an eigenvalue: $E_\lambda(A) = \{0\}$ ($\dim E_\lambda = 0$)

Q: What is $\dim E_\lambda$ when λ is an eigenvalue?

Names: $\dim E_\lambda$ = the geometric multiplicity of λ, = geom mult(λ)

$\text{mult}(\lambda, P_A)$ = the algebraic ____________ . = alg mult(λ)

$\longrightarrow$ $\dim E_\lambda \leq \text{mult}(\lambda, P_A)$ (THM)

$1 \leq \dim E_\lambda$, $\quad 1 \leq \text{mult}(\lambda, P_A)$

Earlier examples:

#	Matrix A	P_A	Eigenvalues	$\dim E_\lambda$	$\text{mult}(\lambda, P_A)$
1	$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$	$(\lambda+1)^2$	$\lambda = -1$	1	2
2	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$	$(\lambda-2)(\lambda+1)^2$	$\lambda = -1$ $\lambda = 2$	1 1	2 1
3	$\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 8 \\ 0 & 0 & 4 \end{bmatrix}$	$-(\lambda-2)(\lambda+1)(\lambda-4)$	$\lambda = -1$ $\lambda = 2$ $\lambda = 4$	1 1 1	1 1 1

Linear independence for eigenvectors

Theorem: Fix A $n\times n$ matrix an a list of k distinct eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$. Pick one eigenvector $\vec{v}_j \neq \vec{0}$ for each λ_j. ($A\vec{v}_j = \lambda_j \vec{v}_j$)

Then: $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is l.i. in $\mathbb{R}^n$.

Why? • $k=1$ $\quad \vec{v}_1 \neq \vec{0}$ so it's l.i.

• Assume $k>1$. We argue by contradiction, assuming S is l.d.

After reordering the vectors we can find $2 \leq m \leq k$ with

$$\{\vec{v}_1, \ldots, \vec{v}_{m-1}\} \text{ l.i. } \quad \& \quad \{\vec{v}_1, \ldots, \vec{v}_m\} \text{ l.d}$$

Write: nontrivial relation, so $a_m \neq 0$

$$a_1\vec{v}_1 + \cdots + a_m\vec{v}_m = \vec{0} \quad (1)$$

$$A(a_1\vec{v}_1 + \cdots + a_m\vec{v}_m) = A\vec{0}$$

$$a_1 A\vec{v}_1 + \cdots + a_m A\vec{v}_m = \vec{0}$$

$$a_1\lambda_1\vec{v}_1 + \cdots + a_m\lambda_m\vec{v}_m = 0 \quad (2)$$

$(2) - \lambda_m (1)$:

$$a_1(\lambda_1-\lambda_m)\vec{v}_1 + \cdots + a_{m-1}(\lambda_m - \lambda_{m-1})\vec{v}_{m-1} = \vec{0}$$

$\{\vec{v}_1, \ldots, \vec{v}_{m-1}\}$ is li, so

$$\left.\begin{aligned} a_1(\lambda_1-\lambda_m) &= 0 \\ &\vdots \\ a_{m-1}(\lambda_m-\lambda_{m-1}) &= 0 \end{aligned}\right\} \rightsquigarrow a_1 = \cdots = a_{m-1} = 0$$

all $\neq 0$

contradicts (1)

Conclude: S is l.i.

Defective Matrices

- Fix A $n\times n$ matrix. We are interested in situations where:

(*) $\boxed{\text{alg mult}(\lambda) = \text{geom mult}(\lambda) \text{ for all } \lambda \text{ eigenvalues of } A}$

Def: We say A is a defective matrix when (*) fails. More precisely, A has a (real or complex) eigenvalue with alg mult $(\lambda) >$ geom mult(λ)

Theorem: alg mult$(\lambda) \geq$ geom mult (λ) for any eigenvalue of A
(HARD!) (real or complex)

Consequence: If all alg multiplicities are 1, then A is NOT defective. (Ex 3)

Q Why? A If so, $\mathbb{R}^n$ will have a basis B consisting of eigenvectors of A

- Then, $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $T(\vec{v}) = A\vec{v}$ will have the following matrix representative:

$$[T]_{BB} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$ (eigenvalues along the diagonal)

Name for this process: "Diagonalizing the matrix A" (§ 4.7)
(not always possible!) (both over real / complex numbers)

ALWAYS possible if alg mult are all 1