Lecture 34: §4.6 Complex Numbers & complex eigenvalues

TODAY's GOAL: Define complex numbers & its use in the EV Problem

2 Motivations for defining complex numbers

① **Algebraic**: Find a new set of numbers enlarging $\mathbb{R}$, so that every polynomial in one variable over $\mathbb{R}$ has a root (Example: x^2+1 has no real roots, but 2 complex roots)

② **Geometric**: Define a multiplication on $\mathbb{R}^2$:
$$\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(\vec{u}, \vec{v}) \longmapsto \vec{u} \odot \vec{v}$$
satisfying same nice properties as mult. in $\mathbb{R}$:

(A) Commutative:

(B) Associative:

(C) Distributive

(D) Neutral Element:

(E) Inverses:

(F) View $\mathbb{R}$ in $\mathbb{R}^2$

(A) Commutative : $\vec{u} \odot \vec{v} = \vec{v} \odot \vec{u}$

(B) Associative : $\vec{u} \odot (\vec{v} \odot \vec{w}) = (\vec{u} \odot \vec{v}) \odot \vec{w}$

(C) Distributive $\vec{u} \odot (\vec{v} + \vec{w}) = \vec{u} \odot \vec{v} + \vec{u} \odot \vec{w}$

(D) Neutral Element: a vector $\vec{e}$ in $\mathbb{R}^2$ with $\vec{u} \odot \vec{e} = \vec{e} \odot \vec{u} = \vec{u}$ for all $\vec{u}$

(E) Inverses : given $\vec{u}$ in $\mathbb{R}^2$ we can find $\vec{v}$ in $\mathbb{R}^2$ with $\vec{u} \odot \vec{v} = \vec{v} \odot \vec{u} = \vec{e}$

(F) View $\mathbb{R}$ in $\mathbb{R}^2$ via $a \longleftrightarrow (a, 0)$ & $\odot$ extends usual mult. in $\mathbb{R}$

Obs: These properties uniquely determine $\odot$ = mult. on $\mathbb{R}^2$.

Need to define: $\begin{bmatrix} x \\ 0 \end{bmatrix} \odot \begin{bmatrix} 0 \\ y \end{bmatrix}$ & $\begin{bmatrix} 0 \\ y_1 \end{bmatrix} \odot \begin{bmatrix} 0 \\ y_2 \end{bmatrix}$

Remember: wants to solve $x^2 + 1 = 0$

Proposal: $\begin{bmatrix} x \\ 0 \end{bmatrix} \odot \begin{bmatrix} 0 \\ y \end{bmatrix} =$ & $\begin{bmatrix} 0 \\ y_1 \end{bmatrix} \odot \begin{bmatrix} 0 \\ y_2 \end{bmatrix} =$

Complex Numbers

<u>Def</u>: A complex number z is given by a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$, written as

$$\boxed{z = a + ib} \qquad \text{(i is a placeholder)}$$

<u>Names</u>: $a = \mathrm{Re}(z)$ = "Real part of z"

$b = \mathrm{Im}(z)$ = "Imaginary part of z"

- $a + ib = c + id$ means $a = c$ & $b = d$.

<u>Properties</u>:

① $(a+ib)(c+id) =$

② $i^2 =$

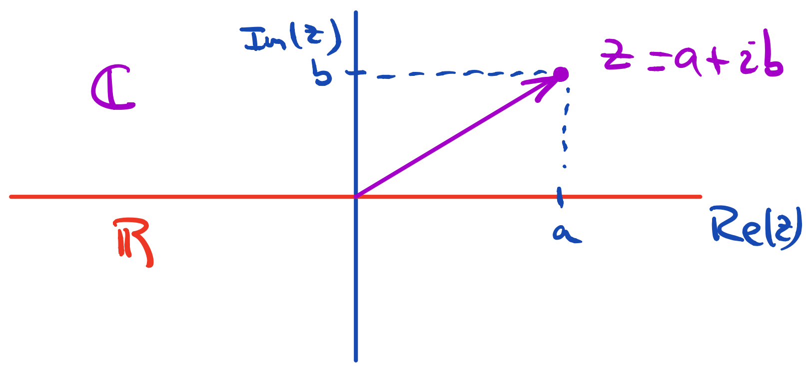
③ real numbers are complex numbers via $a = a + i0$ $(\mathrm{Im}(a) = 0)$

④ $(a+ib) + (c+id) =$

Q How to remember mult rule? $(a+ib)(c+id) = (ac-bd) + i(ad+bc)$

A Use distributive law & $i^2 = -1$

Obs:



$\boxed{\text{New operation: Complex Conjugation}}$

Def: Given $z = a+ib$, its complex conjugate is $\overline{z} = a-ib$ $(= a+i(-b))$

Properties: ① $\overline{z+w} = \overline{z} + \overline{w}$

② $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$

③ $z \cdot \overline{z} = |z|^2$

Consequence:

Application: Use this to write any $\dfrac{a+ib}{c+id}$ with $c+id \neq 0$.

Roots of polynomials in $\mathbb{R}[x]$ & $\mathbb{C}[x]$

Fundamental Theorem of Algebra:

Every non-constant polynomial in 1 variable over $\mathbb{C}$ has a root in $\mathbb{C}$

("$\mathbb{C}$ is algebraically closed")

Example: Quadratic Polynomials

$P(x) = ax^2 + bx + c \qquad a, b, c$ in $\mathbb{C}$, $a \neq 0$

Roots: $= \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ in $\mathbb{C}$

Q: What about polynomials with real coefficients?

A: $\boxed{\begin{array}{l} \text{2 types of roots : (1) real roots} \\ \qquad\qquad\qquad\quad \text{(2) complex roots : they come in conjugate pairs!} \end{array}}$

Why? $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ $\quad a_0, a_1, \ldots, a_n$ in $\mathbb{R}$

- If $\alpha = a + ib$ is a root, then $\bar{\alpha} = a - ib$ is also a root