Lecture 34: §4.6 Complex Numbers & complex eigenvalues

TODAY's GOAL: Define complex numbers & its use in the EV Problem

2 Motivations for defining complex numbers

① **Algebraic**: Find a new set of numbers enlarging $\mathbb{R}$, so that every polynomial in one variable over $\mathbb{R}$ has a root (Example: x^2+1 has no real roots, but 2 complex roots)

② **Geometric**: Define a multiplication on $\mathbb{R}^2$:
$$\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(\vec{u}, \vec{v}) \longmapsto \underbrace{\vec{u} \odot \vec{v}}_{\text{a vector in } \mathbb{R}^2}$$

satisfying same nice properties as mult. in $\mathbb{R}$:

(A) Commutative: $\vec{u} \odot \vec{v} = \vec{v} \odot \vec{u}$

(B) Associative: $\vec{u} \odot (\vec{v} \odot \vec{w}) = (\vec{u} \odot \vec{v}) \odot \vec{w}$

(C) Distributive: $\vec{u} \odot (\vec{v} + \vec{w}) = \vec{u} \odot \vec{v} + \vec{u} \odot \vec{w}$

(D) Neutral Element: a vector $\vec{e}$ in $\mathbb{R}^2$ with $\vec{u} \odot \vec{e} = \vec{e} \odot \vec{u} = \vec{u}$ for all $\vec{u}$

(E) Inverses: given $\vec{u}$ in $\mathbb{R}^2$ we can find $\vec{v}$ in $\mathbb{R}^2$ with $\vec{u} \odot \vec{v} = \vec{v} \odot \vec{u} = \vec{e}$

(F) View $\mathbb{R}$ in $\mathbb{R}^2$ via $a \longleftrightarrow \begin{bmatrix} a \\ 0 \end{bmatrix}$ & $\odot$ extends usual mult. in $\mathbb{R}$

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(F) View $\mathbb{R}$ in $\mathbb{R}^2$ via $a \longleftrightarrow (a,0)$ & $\odot$ extends usual mult. in $\mathbb{R}$

Obs: These properties uniquely determine $\odot$ = mult. on $\mathbb{R}^2$.

- e has to be $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by (F) & (D) (uniqueness of e & role of 1 in $\mathbb{R}$)
- (C) $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \leadsto \begin{bmatrix} a \\ b \end{bmatrix} \odot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \odot \left(\begin{bmatrix} c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \end{bmatrix} \right) =$

$$= \begin{bmatrix} a \\ b \end{bmatrix} \odot \begin{bmatrix} c \\ 0 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} \odot \begin{bmatrix} 0 \\ d \end{bmatrix} = \left(\begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \right) \odot \begin{bmatrix} c \\ 0 \end{bmatrix} + \left(\begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \right) \odot \begin{bmatrix} 0 \\ d \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} a \\ 0 \end{bmatrix} \odot \begin{bmatrix} c \\ 0 \end{bmatrix}}_{\overset{(E)}{=} \begin{bmatrix} ac \\ 0 \end{bmatrix}} + \underbrace{\begin{bmatrix} 0 \\ b \end{bmatrix} \odot \begin{bmatrix} c \\ 0 \end{bmatrix}}_{\overset{(A)}{=} \begin{bmatrix} c \\ 0 \end{bmatrix} \odot \begin{bmatrix} 0 \\ b \end{bmatrix}} + \begin{bmatrix} 0 \\ b \end{bmatrix} \odot \begin{bmatrix} 0 \\ d \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix} \odot \begin{bmatrix} 0 \\ d \end{bmatrix}$$

$\leadsto$ Only need to determine $\begin{bmatrix} x \\ 0 \end{bmatrix} \odot \begin{bmatrix} 0 \\ y \end{bmatrix}$ & $\begin{bmatrix} 0 \\ y_1 \end{bmatrix} \odot \begin{bmatrix} 0 \\ y_2 \end{bmatrix}$

Need to define. $\begin{bmatrix} x \\ 0 \end{bmatrix} \odot \begin{bmatrix} 0 \\ y \end{bmatrix}$ & $\begin{bmatrix} 0 \\ y_1 \end{bmatrix} \odot \begin{bmatrix} 0 \\ y_2 \end{bmatrix}$

Remember: wants to solve $x^2+1=0 \quad \rightsquigarrow \quad \left\{\begin{array}{l} \begin{bmatrix} x \\ 0 \end{bmatrix} \odot \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \\ \begin{bmatrix} x \\ 0 \end{bmatrix} \odot \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{array}\right.$

Proposal: $\begin{bmatrix} x \\ 0 \end{bmatrix} \odot \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ xy \end{bmatrix}$ & $\begin{bmatrix} 0 \\ y_1 \end{bmatrix} \odot \begin{bmatrix} 0 \\ y_2 \end{bmatrix} = \begin{bmatrix} -y_1y_2 \\ 0 \end{bmatrix}$

Check Assoc. (1) $\left(\begin{bmatrix} x \\ 0 \end{bmatrix} \odot \begin{bmatrix} 0 \\ y \end{bmatrix}\right) \odot \begin{bmatrix} 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ xy \end{bmatrix} \odot \begin{bmatrix} 0 \\ z \end{bmatrix} = \begin{bmatrix} -xyz \\ 0 \end{bmatrix}$

$$\begin{bmatrix} x \\ 0 \end{bmatrix} \odot \left(\begin{bmatrix} 0 \\ y \end{bmatrix} \odot \begin{bmatrix} 0 \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix} \odot \begin{bmatrix} -yz \\ 0 \end{bmatrix} = \begin{bmatrix} -xyz \\ 0 \end{bmatrix}$$

(2) $\left(\begin{bmatrix} x \\ 0 \end{bmatrix} \odot \begin{bmatrix} 0 \\ y \end{bmatrix}\right) \odot \begin{bmatrix} z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ xy \end{bmatrix} \odot \begin{bmatrix} z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ xyz \end{bmatrix}$

$$\begin{bmatrix} x \\ 0 \end{bmatrix} \odot \left(\begin{bmatrix} 0 \\ y \end{bmatrix} \odot \begin{bmatrix} z \\ 0 \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix} \odot \left(\begin{bmatrix} z \\ 0 \end{bmatrix} \odot \begin{bmatrix} 0 \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix} \odot \begin{bmatrix} 0 \\ yz \end{bmatrix} = \begin{bmatrix} 0 \\ xyz \end{bmatrix}$$

(3) $\left(\begin{bmatrix} 0 \\ x \end{bmatrix} \odot \begin{bmatrix} 0 \\ y \end{bmatrix}\right) \odot \begin{bmatrix} 0 \\ z \end{bmatrix} = \begin{bmatrix} -xy \\ 0 \end{bmatrix} \odot \begin{bmatrix} 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -xyz \end{bmatrix}$

$$\begin{bmatrix} 0 \\ x \end{bmatrix} \odot \left(\begin{bmatrix} 0 \\ y \end{bmatrix} \odot \begin{bmatrix} 0 \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x \end{bmatrix} \odot \begin{bmatrix} -yz \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -xyz \end{bmatrix}$$

- This is enough to define $\begin{bmatrix} a \\ b \end{bmatrix} \odot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ 0 \end{bmatrix} \odot \begin{bmatrix} 0 \\ b \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \odot \begin{bmatrix} 0 \\ d \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix} \odot \begin{bmatrix} 0 \\ d \end{bmatrix}$

$$= \begin{bmatrix} ac \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ cb \end{bmatrix} + \begin{bmatrix} -bd \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ ad \end{bmatrix} = \begin{bmatrix} ac-bd \\ ad+bc \end{bmatrix}$$

$\rightsquigarrow$ Satisfies (A) — (F)

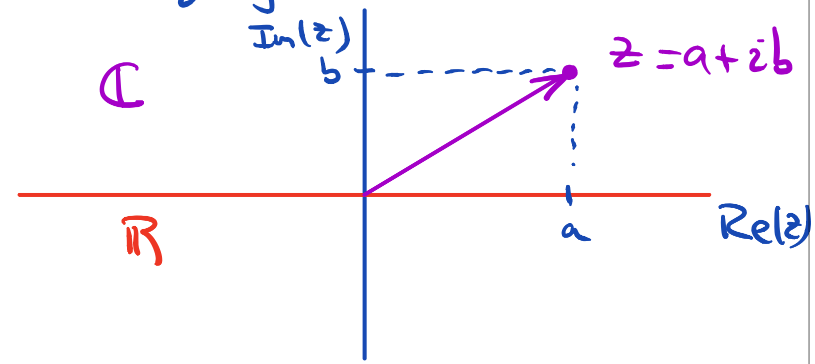
Obs: $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \odot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ so $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ solves $X^2+1=0$ $\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \longleftrightarrow i \text{ in } \mathbb{C}\right)$

Complex Numbers

Def: A complex number z is given by a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$, written as

$$\boxed{z = a + i\,b}$$

(i is a placeholder)



Names: $a = \mathrm{Re}(z)$ = "Real part of z"

$b = \mathrm{Im}(z)$ = "Imaginary part of z"

- $a + ib = c + id$ means $a = c$ & $b = d$.

Properties:

1. $(a+ib)(c+id) = \begin{bmatrix} a \\ b \end{bmatrix} \odot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac - bd \\ ad + bc \end{bmatrix} = (ac-bd) + i(ad-bc)$
2. $i^2 = -1$, so it's a solution to $x^2 + 1 = 0$
3. real numbers are complex numbers via $a = a + i0$ $(\mathrm{Im}(a) = 0)$
4. $(a+ib) + (c+id) = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix} = (a+c) + i(b+d)$

Ex: $z = (1+i)$, $w = (2+i3)$ $\rightsquigarrow$

$z \cdot w = (1+i)(2+i3) = (2-3) + i(3+2) = -1 + i$

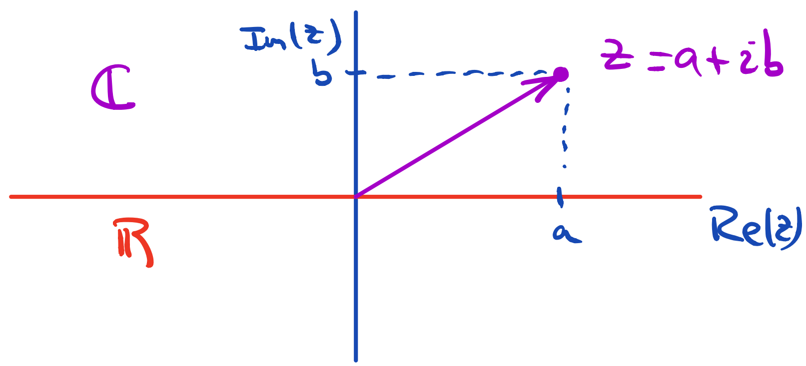
$z + w = (1+i) + (2+i3) = (1+2) + i(1+3) = 3 + 4i$

Q How to remember mult rule? $(a+ib)(c+id) = (ac-bd) + i(ad+bc)$

A Use distributive law & $i^2 = -1$

$$\begin{aligned}(a+ib)(c+id) &= a(c+id) + ib(c+id) \\ &= ac + iad + ibc + \underbrace{i^2}_{=-1} bd \\ &= (ac-bd) + i(ad+bc)\end{aligned}$$

Obs:



View z in $\mathbb{R}^2$, so it has a magnitude!

$$\boxed{|z| = \sqrt{a^2+b^2} = \sqrt{Re(z)^2 + Im(z)^2}}$$

NAME: Modulus of z

Key property: $|zw| = |z|\,|w|$

Why? $z = a+ib \rightsquigarrow |z| = \sqrt{a^2+b^2}$

$w = c+id \qquad |w| = \sqrt{c^2+d^2}$

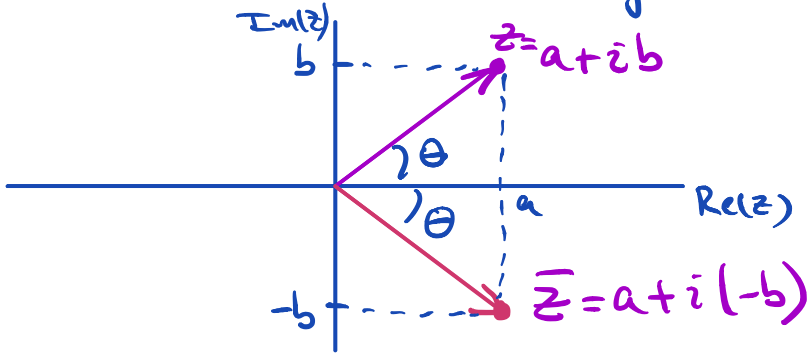
$zw = (ac-bd) + i(ad+bc) \rightsquigarrow |zw| = \sqrt{(ac-bd)^2 + (ad+bc)^2} = \sqrt{(a^2+b^2)(c^2+d^2)}$

New operation: Complex Conjugation

Def: Given $z = a+ib$, its <u>complex conjugate</u> is $\bar{z} = a - ib$ $\quad (= a + i(-b))$

In particular: $z = \bar{z}$ if & only if $b = -b$ (ie, $b = 0$, so z is in $\mathbb{R}$)

<u>Visually</u>:



mirror image about real axis

<u>Properties</u>: ① $\overline{z+w} = \bar{z} + \bar{w}$

<u>Why</u>? $z = a+ib$ $\qquad \overline{z+w} = \overline{(a+c)+i(b+d)} = a+c-i(b+d)$

$w = c+id$ $\qquad \bar{z}+\bar{w} = (a-ib)+(c-id) = a+c-i(b+d)$ ✓

② $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

<u>Why</u>? $\overline{z \cdot w} = \overline{(a+ib)(c+id)} = \overline{(ac-bd)+i(ad+bc)} = (ac-bd) - i(ad+bc)$

$\bar{z} \cdot \bar{w} = (a-ib)(c-id) = (ac-bd) + i(-ad-bc) = ac-bd - i(ad+bc)$ ✓

③ $z \cdot \bar{z} = |z|^2$

<u>Why</u>? $z \cdot \bar{z} = (a+ib)(a-ib) = a^2 - (-b^2) + i(-ab+ab) = a^2+b^2 = |z|^2$ ✓

Consequence: if $z \neq 0$, we have either $\mathrm{Re}(z) \neq 0$ or $\mathrm{Im}(z) \neq 0$, so $|z|^2 > 0$

From $z \cdot \bar{z} = |z|^2$ we conclude z is invertible & $\boxed{z^{-1} = \dfrac{\bar{z}}{|z|^2}}$

Application: Use this to write any $\dfrac{a+ib}{c+id}$ with $c+id \neq 0$.

How? Multiply & divide by $\overline{c+id} = c-id$

$$\frac{a+ib}{c+id} \cdot \frac{c-id}{c-id} = \frac{(a+ib)(c-id)}{|c+id|^2} = \frac{ac+bd}{c^2+d^2} + i\,\frac{(-ad+bc)}{c^2+d^2}.$$

EXAMPLE 1 $z = 1+i$ $\quad \bar{z} = 1-i$ $\quad |z|^2 = 1^2+1^2 = 2 > 0$

so $z^{-1} = \dfrac{1-i}{2} = \dfrac{1}{2} - \dfrac{i}{2}$ $\qquad$ (Check: $(1+i)\left(\frac{1}{2} - \frac{i}{2}\right) = 1$)

EXAMPLE 2 $\dfrac{2+i}{1+i} = \dfrac{2+i}{1-i}\,\dfrac{1-i}{1-i} = \dfrac{(2+1)}{2} + i\,\dfrac{(-2+1)}{2} = \dfrac{3}{2} - \dfrac{i}{2}$

EXAMPLE 3 $z = 1 - i3 \rightsquigarrow \bar{z} = 1 + i3$

$w = 2 + i4 \rightsquigarrow \bar{w} = 2 - i4$

$\Big\} \rightsquigarrow \bar{z}\,\bar{w} = (2+12) + i(-4+6) = 14 + i2$

$z + w = 3 + i \rightsquigarrow \overline{z+w} = 3 - i = \bar{z} + \bar{w}$

$\overline{z \cdot w} = \overline{2 + 12 + i(4-6)} = 14 + i2 = \bar{z}\,\bar{w}.$

$\boxed{\text{Roots of polynomials in } \mathbb{R}[x] \text{ \& } \mathbb{C}[x]}$

Fundamental Theorem of Algebra:

Every non-constant polynomial in 1 variable over $\mathbb{C}$ has a root in $\mathbb{C}$

("$\mathbb{C}$ is algebraically closed")

Example: Quadratic Polynomials

$$P(x) = ax^2+bx+c \qquad a,b,c \text{ in } \mathbb{C}, \quad a\neq 0$$

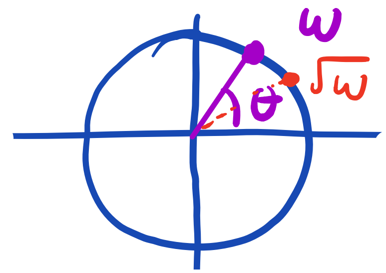
Roots: $$= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \quad \text{in } \mathbb{C}$$

Q: What does $\sqrt{\ }$ mean over $\mathbb{C}$?

Ex: $\sqrt{-1} = i$, $\quad \sqrt{-4} = i2$

In general: $$z = |z| \underbrace{\frac{z}{|z|}}_{=w \text{ has modulus } 1}$$

$$\rightsquigarrow \sqrt{z} = \underbrace{\sqrt{|z|}}_{\text{in } \mathbb{R}_{>0}} \; \underbrace{\sqrt{w}}_{\text{modulus } 1}$$



$|w|=1$ so it lies in the unit circle

$\rightsquigarrow w = \cos\theta + i \sin\theta \qquad 0\leq\theta<2\pi$

$$\rightsquigarrow \boxed{\sqrt{w} = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}$$

Q: What about polynomials with real coefficients?

A: $\boxed{\text{2 types of roots : (1) real roots; (2) complex roots : they come in conjugate pairs!}}$

Why? $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad a_0, a_1, \ldots, a_n$ in $\mathbb{R}$

- If $\alpha = a + ib$ is a root, then $\bar{\alpha} = a - ib$ is also a root

$$f(\bar{\alpha}) = a_n(\bar{\alpha})^n + a_{n-1}(\bar{\alpha})^{n-1} + \cdots + a_1\bar{\alpha} + a_0$$

$$= \bar{a}_n(\bar{\alpha})^n + \bar{a}_{n-1}(\bar{\alpha})^{n-1} + \cdots + \bar{a}_1\bar{\alpha} + \bar{a}_0$$

a's in $\mathbb{R}$

$$\underset{\text{Props}}{=} \overline{(a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0)} = \bar{0} = 0.$$

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -2 & 1 \end{bmatrix} \rightsquigarrow P_A(\lambda) = \det(A - \lambda I_2) = \det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & -2 & 1-\lambda \end{bmatrix}\right)$

$= (\lambda-1)\left((3-\lambda)(1-\lambda)+2\right) = (\lambda^2 - 4\lambda + 5)(\lambda - 1)$

Roots: $\dfrac{-(-4) \pm \sqrt{-4}}{2} = \dfrac{4 \pm 2\sqrt{-1}}{2} = \dfrac{4 \pm i2}{2} = 2 \pm i$ $\begin{cases} 2+i \\ 2-i \end{cases}$ complex conjugate!

⚠ Eigenspaces for $\lambda = 2 \pm i$: in $\mathbb{C}^3$, not $\mathbb{R}^3$ (next time!)