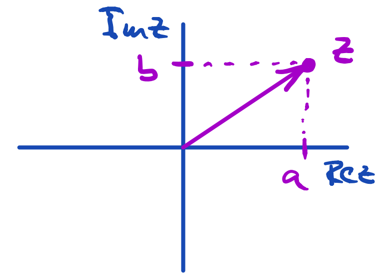
Lecture 35. §4.6 Complex eigenvalues & eigenspaces
Real Symmetric Matrices



Recall

- Complex numbers ($\mathbb{C}$) $\quad z = a + ib \quad$ $a = Re(z)$ in $\mathbb{R}$, $\quad b = Im(z)$ in $\mathbb{R}$
- i satisfies $i^2 = -1$
- **Addition**: $(a+ib)+(c+id) = (a+c) + i(b+d)$
- **Multiplication**: $(a+ib)(c+id) = (ac-bd) + i(ad+bc)$
- **Complex conjugation**: $z = a+ib \rightsquigarrow \overline{z} = a - ib$

  Properties ① $\overline{zw} = \overline{z}\cdot\overline{w}$, ② $\overline{z+w} = \overline{z} + \overline{w}$
- **Modulus**: $|z| = \sqrt{Re(z)^2 + Im z^2}$ $\rightsquigarrow$ ① $z\cdot\overline{z} = |z|^2$ ② $|zw| = |z||w|$

  So $z \neq 0$ has $\boxed{z^{-1} = \frac{\overline{z}}{|z|^2}}$

Fundamental Theorem: Every polynomial in $\mathbb{C}[x]$ of degree ≥ 1 has a root in $\mathbb{C}$

Consequence: Roots of polynomials in $\mathbb{R}[x]$ → real roots / conjugate pairs (same mult.)

$\boxed{\text{Vectors in } \mathbb{C}^n}$

Def: Same as for $\mathbb{R}^n$, but now entries are complex numbers!

Write $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ & impose $v_1, \ldots, v_n$ are in $\mathbb{C}$

- Addition in $\mathbb{C}^n$
- Scalar multiplication =

⚠ Dot Product in $\mathbb{C}^n$: defined using complex conjugation

$\mathbb{C}^n$ is a $\mathbb{C}$-vector space

Structure: $+$ defined entry by entry, scalars = now in $\mathbb{C}$ (instead of $\mathbb{R}$)

① **Closure Properties**: (C1) $\vec{x}, \vec{y}$ in $\mathbb{C}^n$, then $\vec{x}+\vec{y}$ in $\mathbb{C}^n$

(C2) $\vec{x}$ in $\mathbb{C}^n$, then $\alpha\vec{x}$ in $\mathbb{C}^n$

② **Addition Properties**: (A1) $\vec{x}+\vec{y} = \vec{y}+\vec{x}$ (Commutative)

(A2) $\vec{x}+(\vec{y}+\vec{z}) = (\vec{x}+\vec{y})+\vec{z}$ (Associative)

(Neutral element) ← (A3) $\vec{0}$ in $\mathbb{C}^n$ satisfies $\vec{x}+\vec{0} = \vec{0}+\vec{x} = \vec{x}$ for all $\vec{x}$.

(Additive Inverses) ← (A4) Given $\vec{x}$ in $\mathbb{C}^n$ we can find "$-\vec{x}$" in $\mathbb{C}^n$ with $\vec{x}+(-\vec{x}) = \vec{0}$ (here "$-\vec{x}$" $= (-1)\vec{x}$)

③ **Scalar Mult. Properties**: (M1) $\alpha(\beta\vec{x}) = (\alpha\beta)\vec{x}$ (Associative)

(M2) $\alpha(\vec{x}+\vec{y}) = \alpha\vec{x}+\alpha\vec{y}$ (Distributive 1)

(M3) $(\alpha+\beta)\vec{x} = \alpha\vec{x}+\beta\vec{x}$ (——— 2)

(M4) $1\vec{x} = \vec{x}$ for all $\vec{x}$

(A4) follows from (C2)

- Same ideas allow us to define

① $Sp_{\mathbb{C}}(\vec{v_1}, \dots, \vec{v_p}) =$

② $\mathbb{C}$-Linear independence in $\mathbb{C}^n$:

③ Subspaces $\mathbb{V}$ of $\mathbb{C}^n$: 3 properties must hold

(S1)

(S2)

(S3)

④ Basis $B = \{\vec{v_1}, \dots, \vec{v_p}\}$ for a subspace $\mathbb{V}$ of $\mathbb{C}^n$:

Abstract vectors spaces over $\mathbb{C}$

- Same as for $\mathbb{R}$-vector spaces but now scalars are in $\mathbb{C}$

  (10 properties from $\mathbb{C}^n$ model $\mathbb{C}$-abstract vector spaces)

- 2 main examples:

$$\boxed{\text{Eigenvectors in } \mathbb{C}^n}$$

Pick A $n \times n$ matrix with real entries, with λ in $\mathbb{C}$ an eigenvalue

$\rightsquigarrow$ $E_\lambda = \{ \vec{v} \text{ in } \mathbb{C}^n : A\vec{v} = \lambda \vec{v} \} = \mathcal{N}(A - \lambda I_n)$ in $\mathbb{C}^n$

Q: How to find $\mathcal{N}(A - \lambda I_n)$?

(1) $E_{2+i} = \mathcal{N}\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}$

(2) $E_{2-i} = \mathcal{N}\begin{pmatrix} 1+i & 1 \\ -2 & -1+i \end{pmatrix}$

Special case: **real** Symmetric Matrices

Key Theorem: A real $n \times n$ symmetric matrix, then all its eigenvalues are real