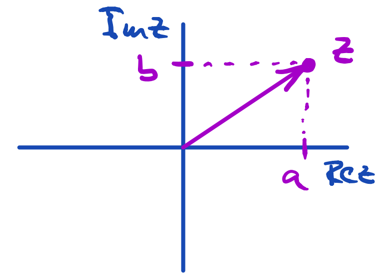
Lecture 35. §4.6 Complex eigenvalues & eigenspaces
Real Symmetric Matrices



Recall

- Complex numbers ($\mathbb{C}$) $\quad z = a + ib \qquad a = \mathrm{Re}(z)$ in $\mathbb{R}$, $\quad b = \mathrm{Im}(z)$ in $\mathbb{R}$
- i satisfies $i^2 = -1$
- **Addition**: $(a+ib)+(c+id) = (a+c) + i(b+d)$
- **Multiplication**: $(a+ib)(c+id) = (ac-bd) + i(ad+bc)$
- **Complex conjugation**: $z = a+ib \rightsquigarrow \overline{z} = a - ib$

  Properties ① $\overline{zw} = \overline{z}\cdot\overline{w}$, ② $\overline{z+w} = \overline{z} + \overline{w}$
- **Modulus**: $|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}\, z^2} \rightsquigarrow$ ① $z\cdot\overline{z} = |z|^2$

  ② $|zw| = |z|\,|w|$

  So $z \neq 0$ has $\boxed{z^{-1} = \dfrac{\overline{z}}{|z|^2}}$

Fundamental Theorem: Every polynomial in $\mathbb{C}[x]$ of degree ≥ 1 has a root in $\mathbb{C}$

Consequence: Roots of polynomials in $\mathbb{R}[x]$ $\nearrow$ real roots, $\searrow$ conjugate pairs (same mult.)

Vectors in $\mathbb{C}^n$

Def: Same as for $\mathbb{R}^n$, but now entries are complex numbers!

Write $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ & impose $v_1, \ldots, v_n$ are in $\mathbb{C}$

- Addition in $\mathbb{C}^n$ is done entry-by-entry
- Scalar multiplication = . scalars in $\mathbb{C}$,
  . also done entry by entry

Example: $\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2+i \\ 4-i \end{bmatrix} \rightsquigarrow \vec{v} + \vec{w} = \begin{bmatrix} 3+i \\ 4 \end{bmatrix}$, $i\vec{v} = \begin{bmatrix} i \\ -1 \end{bmatrix}$

⚠ **Dot Product in $\mathbb{C}^n$**: defined using complex conjugation

$$\vec{v} \cdot \vec{w} = \overline{v_1}\, w_1 + \overline{v_2}\, w_2 + \cdots + \overline{v_n}\, w_n = \overline{\vec{w} \cdot \vec{v}} \quad (\neq \vec{w} \cdot \vec{v})$$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2+i \\ 4-i \end{bmatrix} \rightsquigarrow \vec{v} \cdot \vec{w} = \overline{1}\,(2+i) + \overline{i}\,(4-i)$

$$= 2+i - i(4-i) = (2-1) - i3 = 1 - i3$$

Q: Why? We want $\vec{v} \cdot \vec{v} = |\vec{v}|^2$ to be a non-negative real number!

so $|\vec{v}| = \sqrt{\overline{v_1}\, v_1 + \overline{v_2}\, v_2 + \cdots + \overline{v_n}\, v_n} = \sqrt{\underbrace{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2}_{\geq 0}} \geq 0$

$\boxed{\mathbb{C}^n \text{ is a } \mathbb{C}\text{-vector space}}$

Structure: $+$ defined entry by entry, scalars = now in $\mathbb{C}$ (instead of $\mathbb{R}$)

① **Closure Properties**: (C1) $\vec{x}, \vec{y}$ in $\mathbb{C}^n$, then $\vec{x}+\vec{y}$ in $\mathbb{C}^n$

(C2) $\vec{x}$ in $\mathbb{C}^n$, then $\alpha\vec{x}$ in $\mathbb{C}^n$

② **Addition Properties**: (A1) $\vec{x}+\vec{y} = \vec{y}+\vec{x}$ (Commutative)

(A2) $\vec{x}+(\vec{y}+\vec{z}) = (\vec{x}+\vec{y})+\vec{z}$ (Associative)

(Neutral element) ⟵ (A3) $\vec{0}$ in $\mathbb{C}^n$ satisfies $\vec{x}+\vec{0} = \vec{0}+\vec{x} = \vec{x}$ for all $\vec{x}$.

(Additive Inverses) ⟵ (A4) Given $\vec{x}$ in $\mathbb{C}^n$ we can find "$-\vec{x}$" in $\mathbb{C}^n$ with $\vec{x}+(-\vec{x}) = \vec{0}$ (here "$-\vec{x}$" $= (-1)\vec{x}$)

③ **Scalar Mult. Properties**: (M1) $\alpha(\beta\vec{x}) = (\alpha\beta)\vec{x}$ (Associative)

(M2) $\alpha(\vec{x}+\vec{y}) = \alpha\vec{x}+\alpha\vec{y}$ (Distributive 1)

(M3) $(\alpha+\beta)\vec{x} = \alpha\vec{x}+\beta\vec{x}$ (—— 2)

(A4) follows from (C2)

(M4) $1\vec{x} = \vec{x}$ for all $\vec{x}$

• Same ideas allow us to define

① $Sp_{\mathbb{C}}(\vec{v}_1, \ldots, \vec{v}_p) = \{ \alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p : \alpha_1, \ldots, \alpha_p \text{ in } \mathbb{C} \}$

(span of vectors $\vec{v}_1, \ldots, \vec{v}_p$ in $\mathbb{C}^n$)

② <u>$\mathbb{C}$-Linear independence in $\mathbb{C}^n$</u>:

$\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p = \vec{0}$ in $\mathbb{C}^n$ has only 1 soln $\alpha_1 = \cdots = \alpha_p = 0$

with $\alpha_1, \ldots, \alpha_p$ in $\mathbb{C}$

③ <u>Subspaces $\mathbb{V}$ of $\mathbb{C}^n$</u>: 3 properties must hold

(S1) $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{V}$

(S2) $\vec{v}, \vec{w}$ in $\mathbb{V}$, then $\vec{v} + \vec{w}$ in $\mathbb{V}$

(S3) $\vec{v}$ in $\mathbb{V}$, α in $\mathbb{C}$, then $\alpha \vec{v}$ in $\mathbb{V}$

④ Basis $B = \{\vec{v}_1, \ldots, \vec{v}_p\}$ for a subspace $\mathbb{V}$ of $\mathbb{C}^n$:

• B spans $\mathbb{V}$, ie $\mathbb{V} = Sp_{\mathbb{C}}(\vec{v}_1, \ldots, \vec{v}_p)$

& • B is $\mathbb{C}$-l.i. $\rightsquigarrow$ $\boxed{\#B = \dim_{\mathbb{C}} \mathbb{V}}$

Abstract vector spaces over $\mathbb{C}$

- Same as for $\mathbb{R}$-vector spaces but now scalars are in $\mathbb{C}$

(10 properties from $\mathbb{C}^n$ model $\mathbb{C}$-abstract vector spaces)

- 2 main examples:

① $\mathrm{Mat}_{m\times n}(\mathbb{C})$ = $m\times n$ matrices with entries in $\mathbb{C}$

Addition = entry-by-entry, Scalar mult = entry-by-entry

Ex: $\begin{bmatrix} i & 1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ i & 4 \end{bmatrix} = \begin{bmatrix} i & 2 \\ -1+i & 7 \end{bmatrix}$; $i\begin{bmatrix} i & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & i \\ -i & 3i \end{bmatrix}$

Basis $= \{ E_{ij} : 1\leq i\leq m,\ 1\leq j\leq n \}$ (same as for $\mathrm{Mat}_{m\times n}(\mathbb{R})$)

② $\mathcal{P}_n(\mathbb{C}) = \{ p_{(x)} = a_0 + a_1 x + \cdots + a_n x^n : a_0, a_1, \ldots, a_n \text{ in } \mathbb{C} \}$

- Usual addition & scalar mult
- Ex $(n=2)$: $(1+iX) + (2-2iX^2) = 2+iX-2iX^2$

$i(1+iX) = i - X$

Basis $= \{ 1, X, \ldots, X^n \}$ (same as for $\mathcal{P}_n$ with real coeff)

$\boxed{\text{Eigenvectors in } \mathbb{C}^n}$

Pick A $n \times n$ matrix with real entries, with λ in $\mathbb{C}$ an eigenvalue

$\rightsquigarrow$ $E_\lambda = \{ \vec{v} \text{ in } \mathbb{C}^n : A\vec{v} = \lambda \vec{v} \} = \mathcal{N}(A - \lambda I_n)$ in $\mathbb{C}^n$

Q: How to find $\mathcal{N}(A-\lambda I_n)$?

A: Use Gauss-Jordan to put $A - \lambda I_n$ in REF, but now we are allowed to use complex numbers in the reduction process.

EXAMPLE 1 $\quad A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ $\quad p_A(\lambda) = \det \begin{pmatrix} 3-\lambda & 1 \\ -2 & 1-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 5$

Roots: $\dfrac{-(-4) \pm \sqrt{16 - 4 \cdot 5}}{2} = \dfrac{4 \pm \sqrt{-4}}{2} = \dfrac{4 \pm 2i}{2}$ $\begin{array}{l} \nearrow 2+i \\ \searrow 2-i \end{array}$

$$E_{2+i} = \mathcal{N}\left(A - (2+i) I_2 \right) = \mathcal{N}\begin{pmatrix} 3-(2+i) & 1 \\ -2 & 1-(2+i) \end{pmatrix} = \mathcal{N}\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}$$

$$E_{2-i} = \mathcal{N}\left(A - (2-i) I_2 \right) = \mathcal{N}\begin{pmatrix} 3-(2-i) & 1 \\ -2 & 1-(2-i) \end{pmatrix} = \mathcal{N}\begin{pmatrix} 1+i & 1 \\ -2 & -1+i \end{pmatrix}$$

(1) $E_{2+i} = \mathcal{N}\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}$ (2) $E_{2-i} = \mathcal{N}\begin{pmatrix} 1+i & 1 \\ -2 & -1+i \end{pmatrix}$

(1) $\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix} \xrightarrow[R_1 \to \frac{1}{1-i}R_1]{} \begin{bmatrix} 1 & \boxed{\frac{1}{1-i}} \\ -2 & -1-i \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}+\frac{i}{2} \\ -2 & -1-i \end{bmatrix} \xrightarrow[R_2 \to R_2 + 2R_1]{} \begin{bmatrix} 1 & \frac{1+i}{2} \\ 0 & 0 \end{bmatrix}$ REF

$\frac{1}{1-i} = \frac{1+i}{|1-i|^2}$

$x = -\frac{1+i}{2}\, y$, y free in $\mathbb{C}$ $\rightsquigarrow E_{2+i} = \mathrm{Sp}\left(\begin{bmatrix} -1-i \\ 2 \end{bmatrix}\right)$

(2) $\begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix} \xrightarrow[R_1 \to \frac{1}{1+i}R_1]{} \begin{bmatrix} 1 & \boxed{\frac{1}{1+i}} \\ -2 & -1+i \end{bmatrix} = \begin{bmatrix} 1 & \frac{1-i}{2} \\ -2 & -1+i \end{bmatrix} \xrightarrow[R_2 \to R_2 + 2R_1]{} \begin{bmatrix} 1 & \frac{1-i}{2} \\ 0 & 0 \end{bmatrix}$ REF

$\frac{1-i}{|1+i|^2} = \frac{1}{1+i}$

$x = -\frac{(1-i)}{2}\, y$, y free in $\mathbb{C}$ $\rightsquigarrow E_{2-i} = \mathrm{Sp}\left(\begin{bmatrix} -1+i \\ 2 \end{bmatrix}\right)$

<u>Obs</u>: $\vec{v}$ in $E_{2+i} \rightsquigarrow \overline{\vec{v}}$ in E_{2-i} (bases vectors are complex conjugate!)

<u>Rule</u> A $n\times n$ with real entries & $\lambda, \bar{\lambda}$ complex roots of P_A.

$B = \{\vec{v}_1, \ldots, \vec{v}_s\}$ basis for $E_\lambda \iff \overline{B} = \{\overline{\vec{v}_1}, \ldots, \overline{\vec{v}_s}\}$ basis for $E_{\bar{\lambda}}$

<u>Why?</u> $A\vec{v} = \lambda\vec{v}$ complex conjugate gives $A\overline{\vec{v}} = \overline{A\vec{v}} = \overline{\lambda\vec{v}} = \bar{\lambda}\,\overline{\vec{v}}$

Special case: real Symmetric Matrices

Key Theorem: A real $n\times n$ symmetric matrix, then all its eigenvalues are real

Why? Pick an eigenvalue λ in $\mathbb{C}$ & $\vec{v} \neq \vec{0}$ in $\mathbb{C}^n$ with $A\vec{v} = \lambda\vec{v}$. To show λ in $\mathbb{R}$ $(\lambda = \bar{\lambda})$

- $A\vec{v} = \lambda\vec{v}$ & multiply both sides by $\overline{v^T}$ in $\mathbb{C}^n$ on the left.

- $\overline{v}^T A v = \overline{v}^T \lambda v = \lambda(\vec{v}\cdot\vec{v}) = \boxed{\lambda\|v\|^2}$ (1)

But: $\overline{v}^T(\lambda v) = [\overline{v_1} \;\cdots\; \overline{v_n}]\begin{bmatrix}\lambda v_1\\ \vdots \\ \lambda v_n\end{bmatrix} = [\lambda v_1 \;\cdots\; \lambda v_n]\begin{bmatrix}\overline{v_1}\\ \vdots \\ \overline{v_n}\end{bmatrix} = (\lambda\vec{v})^T\,\overline{\vec{v}}$

- So $\overline{v}^T(\lambda v) = (\lambda\vec{v})^T\,\overline{\vec{v}} = (A\vec{v})^T\,\overline{\vec{v}} = \vec{v}^T A^T\,\overline{\vec{v}} = \vec{v}^T A\,\overline{\vec{v}}$ (A symm)

$= \vec{v}^T(A\overline{\vec{v}}) = \vec{v}^T(\bar{\lambda}\,\overline{\vec{v}}) = \bar{\lambda}\,\vec{v}^T\,\overline{\vec{v}} = \boxed{\bar{\lambda}\,\|\vec{v}\|^2}$ (2)

$\overline{\vec{v}}$ in $E_{\bar{\lambda}}$

So (1) & (2) give $\lambda\|\vec{v}\|^2 = \bar{\lambda}\,\|\vec{v}\|^2$ & $\|\vec{v}\|^2 \neq 0$

so $\lambda = \bar{\lambda}$ $\square$

Example $A = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = A^T$ $\quad P_A(\lambda) = \det\begin{bmatrix}-\lambda & 1\\ 1 & -\lambda\end{bmatrix} = \lambda^2 - 1$
$= (\lambda+1)(\lambda-1)$

$E_1 = N\left(\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\right) = Sp\left(\begin{bmatrix}1\\ 1\end{bmatrix}\right)$, $\quad E_{-1} = N\left(\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right) = Sp\left(\begin{bmatrix}-1\\ 1\end{bmatrix}\right)$