Lecture 36: §4.7 Similarities & Diagonalization

Last time Defined $\mathbb{C}$-vector spaces & focused on 5 examples

(1) $\mathbb{C}^n = \{ \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, v_1, \ldots, v_n \text{ in } \mathbb{C} \}$

$+$ & scalar multiplication (by $\mathbb{C}$): entry-by-entry

(2) $\mathrm{Span}_{\mathbb{C}}(\vec{v_1}, \ldots, \vec{v_p})$ = all $\mathbb{C}$-linear comb. of $\vec{v_1}, \ldots, \vec{v_p}$

$= \{ \alpha_1 \vec{v_1} + \cdots + \alpha_p \vec{v_p} : \alpha_1, \ldots, \alpha_p \text{ in } \mathbb{C} \}$

(3) $\mathrm{Mat}_{m\times n}(\mathbb{C}) = \{ A = (a_{ij})$ of size $m\times n$: a_{ij} in $\mathbb{C} \}$

$+$ & scalar multiplication (by $\mathbb{C}$): entry-by-entry

(4) $\mathcal{N}(A) = \{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{C}^n : A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \} \rightsquigarrow A \sim A'$ REF

Compute A' via Gauss-Jordan using $\mathbb{C}$ in elementary row operations

(5) $\mathcal{P}_n(\mathbb{C}) = \{ a_0 + a_1 x + \cdots + a_n x^n : a_0, \ldots, a_n \text{ in } \mathbb{C} \}$

$+$ & scalar mult: coeff-by-coeff

Similarities

<u>Definition</u> Fix A, C in $\mathrm{Mat}_{n\times n}(\mathbb{C})$. We say A is <u>similar</u> to C (and write $A \sim C$) if we can find S non-singular in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ ($\det(S) \neq 0$ in $\mathbb{C}$) with $C = S^{-1} \cdot A S$.

<u>Motivation:</u> A & C will represent the same $\mathbb{C}$-linear Transformation $T: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ with respect to different bases: B_A & B_C.

Then S will be the matrix recording the change of basis from B_C to B_A

<u>Typical situation</u>: $T(\vec{v}) = A\vec{v}$ so $[T]_{\mathcal{E}\mathcal{E}} = A$ $\mathcal{E} = \{\vec{e_1}, \ldots \vec{e_n}\}$

If $B_C = \{\vec{v_1}, \ldots, \vec{v_n}\}$, then $C = S^{-1} A S$ with $S = [\vec{v_1} \cdots \vec{v_n}]$

Q: Why do we care about similar matrices?

Proposition 1: Similar matrices have the same characteristic polynomials (so same eigenvalues with the same algebraic multiplicity!)

⚠ If $P_A(\lambda) = P_C(\lambda)$, this does NOT mean $A \sim C$.

Q: How are $E_\lambda(C)$ & $E_\lambda(A)$ related when $A \sim C$?

Proposition 2: If $C = S^{-1}AS$ & $\vec{v} \neq \vec{0}$ is in $E_\lambda(C)$, then $\vec{w} = S\vec{v}$ is in $E_\lambda(A)$. Moreover, $\dim E_\lambda(C) = \dim E_\lambda(A)$ &

$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_p\}$ basis for $E_\lambda(C)$ $\qquad$ $\tilde{\mathcal{B}} = \{S\vec{v}_1, \dots, S\vec{v}_p\}$ basis for $E_\lambda(A)$.

Example $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Diagonalization

Def: We say an $n \times n$ real matrix A is <u>diagonalizable</u> over $\mathbb{C}$ if it is similar to a diagonal matrix $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ in $\mathrm{Mat}_{n \times n}(\mathbb{C})$

So $D = S^{-1} A S$ for S in $\mathrm{Mat}_{n \times n}(\mathbb{C})$ nonsingular ($\det S \neq 0$)

Example 1 $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$

Example 2: $A = \begin{bmatrix} 5 & -6 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Real Symmetric Matrices

Diagonalizable over $\mathbb{R}$!

Theorem: Fix A $n \times n$ real symmetric matrix. Then:

① A has only real eigenvalues $\lambda_1, \dots, \lambda_n$

② $\mathbb{R}^n$ has an orthonormal basis B of eigenvectors for A

that is, $B\{\vec{v}_1, \dots, \vec{v}_n\}$ with v_i in $E_{\lambda_i}(A)$. and $v_i \cdot v_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

③ $$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = Q^T A Q \quad \text{with } Q = [\vec{v}_1 \cdots \vec{v}_n]$$