Lecture 36: §4.7 Similarities & Diagonalization

Last Time Defined $\mathbb{C}$-vector spaces & focused on 5 examples

① $\mathbb{C}^n = \{ \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, v_1, \ldots, v_n \text{ in } \mathbb{C} \}$

$+$ & scalar multiplication (by $\mathbb{C}$): entry-by-entry

Basis $B = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$ (for linear comb = use $\mathbb{C}$ coeff)

② $\mathrm{Span}_{\mathbb{C}}(\vec{v}_1, \ldots, \vec{v}_p)$ = all $\mathbb{C}$-linear comb. of $\vec{v}_1, \ldots, \vec{v}_p$

$= \{ \alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p : \alpha_1, \ldots, \alpha_p \text{ in } \mathbb{C} \}$

③ $\mathrm{Mat}_{m \times n}(\mathbb{C}) = \{ A = (a_{ij})$ of size $m \times n$: a_{ij} in $\mathbb{C} \}$

$+$ & scalar multiplication (by $\mathbb{C}$): entry-by-entry

Basis $B = \{ E_{ij} \quad 1 \le i \le m, \; 1 \le j \le n \}$ (again, use $\mathbb{C}$-lin. comb)

④ $\mathcal{N}(A) = \{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{C}^n : A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \}$ $\rightsquigarrow A \sim A'$ REF

Compute A' via Gauss-Jordan using $\mathbb{C}$ in elementary row operations

⑤ $\mathcal{P}_n(\mathbb{C}) = \{ a_0 + a_1 x + \cdots + a_n x^n : a_0, \ldots, a_n \text{ in } \mathbb{C} \}$

$+$ & scalar mult: coeff-by-coeff; Basis $B = \{1, x, \ldots, x^n\}$

Similarities

Definition Fix A, C in $\mathrm{Mat}_{n\times n}(\mathbb{C})$. We say A is similar to C (and write $A \sim C$) if we can find S non-singular in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ ($\det(S) \neq 0$ in $\mathbb{C}$) with $C = S^{-1} A S$.

Motivation: A & C will represent the same $\mathbb{C}$-linear transformation $T: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ with respect to different bases: B_A & B_C.

Then S will be the matrix recording the change of basis from B_C to B_A

Typical situation: $T(\vec{v}) = A\vec{v}$ so $[T]_{EE} = A$ $\quad E = \{\vec{e_1}, \ldots \vec{e_n}\}$

If $B_C = \{\vec{v_1}, \ldots, \vec{v_n}\}$, then $C = S^{-1} A S$ with $S = [\vec{v_1} \cdots \vec{v_n}]$

Example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ $\quad P_A = (1-\lambda)(2-\lambda)$: $\mathcal{N}(A - I_2) = \mathcal{N}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \mathrm{Sp}(e_1)$

$\lambda = 1, 2$ eigenvalues $\quad \mathcal{N}(A - 2I_2) = \mathcal{N}\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{Sp}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $\underbrace{C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{\text{diagonal}}$, $\underbrace{S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{B_C = \text{eigenvectors}}$, $S^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $\quad C = S^{-1} A S$

Q: Why do we care about similar matrices?

Proposition 1: Similar matrices have the same characteristic polynomials (so same eigenvalues with the same algebraic multiplicity!)

Why? Write $C = S^{-1} A S$

$$\boxed{P_C(\lambda)} = \det(S^{-1} A S - \lambda I_n) = \det(S^{-1}(A - \lambda I_n) S)$$
$$= \det(S^{-1}) \det(A - \lambda I_n) \det(S)$$
$$= \frac{1}{\det S} \det(A - \lambda I_n) \det(S) = \boxed{P_A(\lambda)}.$$

So same roots with same multiplicity!

⚠ If $P_A(\lambda) = P_C(\lambda)$, this does NOT mean $A \sim C$.

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $P_A(\lambda) = P_C(\lambda) = (\lambda - 1)^2$

But $S^{-1} A S = S^{-1} I_2 S = I_2 \neq C$ no matter what S is!

Q: How are $E_\lambda(C)$ & $E_\lambda(A)$ related when $A \sim C$?

Proposition 2: If $C = S^{-1}AS$ & $\vec{v} \neq \vec{0}$ is in $E_\lambda(C)$, then $\vec{w} = S\vec{v}$ is in $E_\lambda(A)$. Moreover, $\dim E_\lambda(C) = \dim E_\lambda(A)$ &

$$\underset{\text{basis for } E_\lambda(C)}{B = \{\vec{v}_1, \ldots, \vec{v}_p\}} \underset{S^{-1}\cdot}{\overset{S\cdot}{\rightleftarrows}} \underset{\text{basis for } E_\lambda(A).}{\tilde{B} = \{S\vec{v}_1, \ldots, S\vec{v}_p\}}$$

Proof: Know $C\vec{v} = \lambda\vec{v} \rightsquigarrow S^{-1}AS\vec{v} = \lambda\vec{v}$

$A(S\vec{v}) = S\lambda\vec{v} = \lambda(S\vec{v})$

But S invertible forces $\vec{w} = S\vec{v} \neq \vec{0}$. & $\vec{w}$ in $E_\lambda(A)$

- Now, pick a basis B for $E_\lambda(C)$, then $\tilde{B} = \{w_1 = S\vec{v}_1, \ldots, \vec{w}_p = S\vec{v}_p\}$ consists of eigenvectors for A. Claim: $\tilde{B}$ is li

$\vec{0} = \alpha_1\vec{w}_1 + \cdots + \alpha_p\vec{w}_p = \alpha_1 S\vec{v}_1 + \cdots + \alpha_p S\vec{v}_p$.

$= S(\alpha_1\vec{v}_1 + \cdots + \alpha_p\vec{v}_p)$ & S nonsing, so $\alpha_1\vec{v}_1 + \cdots + \alpha_p\vec{v}_p = \vec{0}$

B is li so $\alpha_1 = \cdots = \alpha_p = 0$, meaning $\tilde{B}$ is li.

- Thus, $\dim E_\lambda(A) \geq p = \dim E_\lambda(C)$. $\longrightarrow$ $\dim E_\lambda(C) = \dim E_\lambda(A)$
- Since $A = S^{-1}CS$, reverse the roles of C & A to get $\dim E_\lambda(C) \geq \dim E_\lambda(A)$

Example $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Claim: $A \sim C$

$$P_A(\lambda) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda-1)(\lambda+1) = P_C(\lambda)$$

$$E_1(A) = \mathcal{N}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = Sp\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$E_{-1}(A) = \mathcal{N}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = Sp\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

If $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ then $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

gives $[T]_{BB} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = C$, $[T]_{EE} = A$. So $C = S^{-1} A S$

with $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \rightsquigarrow S^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

Summary:
- Showed $A \sim C$ by explicitly computing S.
- Used eigenvectors because C was a diagonal matrix
- This is diagonalization in a nutshell.

Diagonalization

Def: We say an $n\times n$ real matrix A is <u>diagonalizable</u> over $\mathbb{C}$ if it is similar to a diagonal matrix $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ in $\mathrm{Mat}_{n\times n}(\mathbb{C})$

So $D = S^{-1} A S$ for S in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ nonsingular ($\det S \neq 0$)

Observe: $P_A(\lambda) = P_D(\lambda) = (\lambda - d_1) \cdots (\lambda - d_n)$

so the eigenvalues of A are $d_1, \ldots d_n$ (counted with multiplicity, ie possible repetitions!)

$S = [\vec{v}_1 \; \cdots \; \vec{v}_n]$ involves a basis for $\mathbb{C}^n$ of eigenvectors of A

with $\vec{v}_1$ in $E_{d_1}(A)$
$\vdots$
$\vec{v}_n$ in $E_{d_n}(A)$

⚠ **Order** of the columns of S is **consistent** with the entries of D

Next Time: Necessary and sufficient conditions to diagonalize A.

($\dim_{\mathbb{C}} E_\lambda(A) = \mathrm{mult}(\lambda, P_A)$ for each λ eigenvalue of A)

Example ① $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ $\quad p_A(\lambda) = \det\begin{pmatrix} 5-\lambda & -6 \\ 3 & -4-\lambda \end{pmatrix} = (5-\lambda)(-4-\lambda) + 18$

$$= \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$$

- 2 eigenvalues $\lambda = -1$ & 2 of multiplicity 1.

know $1 \leq \dim E_2(A) \leq 1$ & $1 \leq \dim E_{-1}(A) \leq 1$ so both dims are 1.

- $1+1 = 2 = \dim \mathbb{R}^2$ so $\mathbb{R}^2$ has a basis of eigenvectors of A

$$E_{-1}(A) = \mathcal{N}(A + I_2) = \mathcal{N}\begin{pmatrix} 6 & -6 \\ 3 & -3 \end{pmatrix} = \mathcal{N}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \mathrm{Sp}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$E_2(A) = \mathcal{N}(A - 2I_2) = \mathcal{N}\begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix} = \mathcal{N}\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} = \mathrm{Sp}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad S^{-1} = \frac{1}{1-2}\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

basis for E_{-1} basis for E_2

$$D = S^{-1} A S \quad \text{so} \quad A = S D S^{-1}$$

Example ② $A = \begin{bmatrix} 5 & -6 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $\quad p_A(\lambda) = -(\lambda-2)(\lambda+1)(\lambda-5)$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad S = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \quad S^{-1} = \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Real Symmetric Matrices

Diagonalizable over $\mathbb{R}$!

<u>Theorem</u>: Fix A $n\times n$ real symmetric matrix. Then:

① A has only real eigenvalues $\lambda_1, \dots, \lambda_n$ (possibly repeated!) L35

② $\mathbb{R}^n$ has an orthonormal basis B of eigenvectors for A

that is, $B \{\vec{v_1}, \dots, \vec{v_n}\}$ with v_i in $E_{\lambda_i}(A)$. and $v_i \cdot v_j = \begin{cases} 0 & i\neq j \\ 1 & i=j \end{cases}$

③ $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = Q^T A Q$ with $Q = [\vec{v_1} \cdots \vec{v_n}]$

($Q^{-1} = Q^T$ because B is orthonormal!)

<u>Example</u>: $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = A^T$ $\quad P_A(\lambda) = \det\begin{pmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1$
$= \lambda(\lambda-2)$

$E_0 = \mathcal{N}(A) = \mathcal{N}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = Sp\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = Sp\left(\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right)$ → norm = 1

$E_2 = \mathcal{N}(A - 2I_2) = \mathcal{N}\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} = \mathcal{N}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = Sp\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = Sp\left(\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\right)$

Check: $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = 0$ Take $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ $\quad C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.