Lecture 37: §4.7 Similarities & Diagonalization

<u>Recall</u>: An $n \times n$ real matrix A is <u>diagonalizable</u> over $\mathbb{C}$ if $A \sim D$

where $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ is a diagonal matrix in $Mat_{n\times n}(\mathbb{C})$

So $D = S^{-1} A S$ for S in $Mat_{n\times n}(\mathbb{C})$ nonsingular ($\det S \neq 0$)

- $\{d_1, \ldots, d_n\}$ are the eigenvalues of A (with possible repetitions)
- If $S = [\vec{v}_1, \ldots \vec{v}_n]$ then
$$A\vec{v}_1 = d_1 \vec{v}_1$$
$$A\vec{v}_2 = d_2 \vec{v}_2$$
$$\vdots$$
$$A\vec{v}_n = d_n \vec{v}_n$$
& $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a bases for $\mathbb{C}^n$
(basis of eigenvectors!)
- S invertible so $S^{-1} = \dfrac{Cof(S)^T}{\det(S)}$, also $(S \mid I_n) \overset{GJ}{\sim} (I_n \mid S^{-1})$
$\mathbb{C}$ row-operations

<u>Conclusion</u>: A is diagonalizable over $\mathbb{C}$ <u>means</u> $\mathbb{C}^n$ has a basis of eigenvectors

Real Symmetric Matrices

Diagonalizable over $\mathbb{R}$!

Theorem: Fix A $n \times n$ real symmetric matrix. Then:

① A has only real eigenvalues $\lambda_1, \dots, \lambda_n$ (possibly repeated!) L35

② $\mathbb{R}^n$ has an orthonormal basis B of eigenvectors for A

that is, $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ with v_i in $E_{\lambda_i}(A)$. and $v_i \cdot v_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

③ $$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = Q^T A Q \quad \text{with } Q = [\vec{v}_1 \cdots \vec{v}_n]$$

($Q^{-1} = Q^T$ because B is orthonormal!)

$\boxed{\text{Why diagonalize?}}$

A: We can perform power operations very fast!

Proposition: ① If $D = S^{-1}AS$ $(A \sim D)$, then $A^k = SD^kS^{-1}$ for all $k = 1, 2, \ldots$

② If $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$, then $D^k = \begin{pmatrix} d_1^k & & 0 \\ & \ddots & \\ 0 & & d_n^k \end{pmatrix}$ for all $k = 1, 2, \ldots$

If D is invertible, the same formulas work for $k = -1, -2, \ldots$

__Example__ (last time) $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ $p_A(\lambda) = (\lambda-2)(\lambda+1)$ $E_{-1} = Sp\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$E_2 = Sp\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$

Diagonalization over $\mathbb{R}$ vs over $\mathbb{C}$

Fix A an $n \times n$ matrix with entries in $\mathbb{R}$.

Theorem: ① A is diagonalizable over $\mathbb{R}$ if, and only if, all eigenvalues of A are <u>real</u> & $\mathbb{R}^n$ has a basis of eigenvectors for A.

② A is diagonalizable over $\mathbb{C}$ if, and only if, $\mathbb{C}^n$ has a basis of eigenvectors for A.

- Diagonalization of A over $\mathbb{C}$ = basis for $\mathbb{C}^n$ with eigenvectors

$\boxed{\text{Diagonalizable} = \text{non-defective}}$ A in $\mathrm{Mat}_{n\times n}(\mathbb{R})$

[L33]

Proposition: Pick $\{\lambda_1, \ldots, \lambda_p\}$ distinct eigenvalues for A & $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ with $\vec{v}_i \neq \vec{0}$ & $A\vec{v}_i = \lambda_i \vec{v}_i$. Then S is li

Consequence: If $\{\lambda_1, \ldots, \lambda_r\}$ are all the eigenvalues of A (in $\mathbb{C}$) & $\dim_{\mathbb{C}} E_{\lambda_i}(A) = \mathrm{mult}(\lambda_i, P_A)$ for $i = 1, \ldots, r$, then $\mathbb{C}^n$ has a basis of eigenvectors.

Example:

$$A = \begin{bmatrix} 25 & -8 & 30 \\ 24 & -7 & 30 \\ -12 & 4 & -14 \end{bmatrix} \rightsquigarrow P_A(\lambda) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2$$

$$= -(\lambda - 1)^2 (\lambda - 2)$$