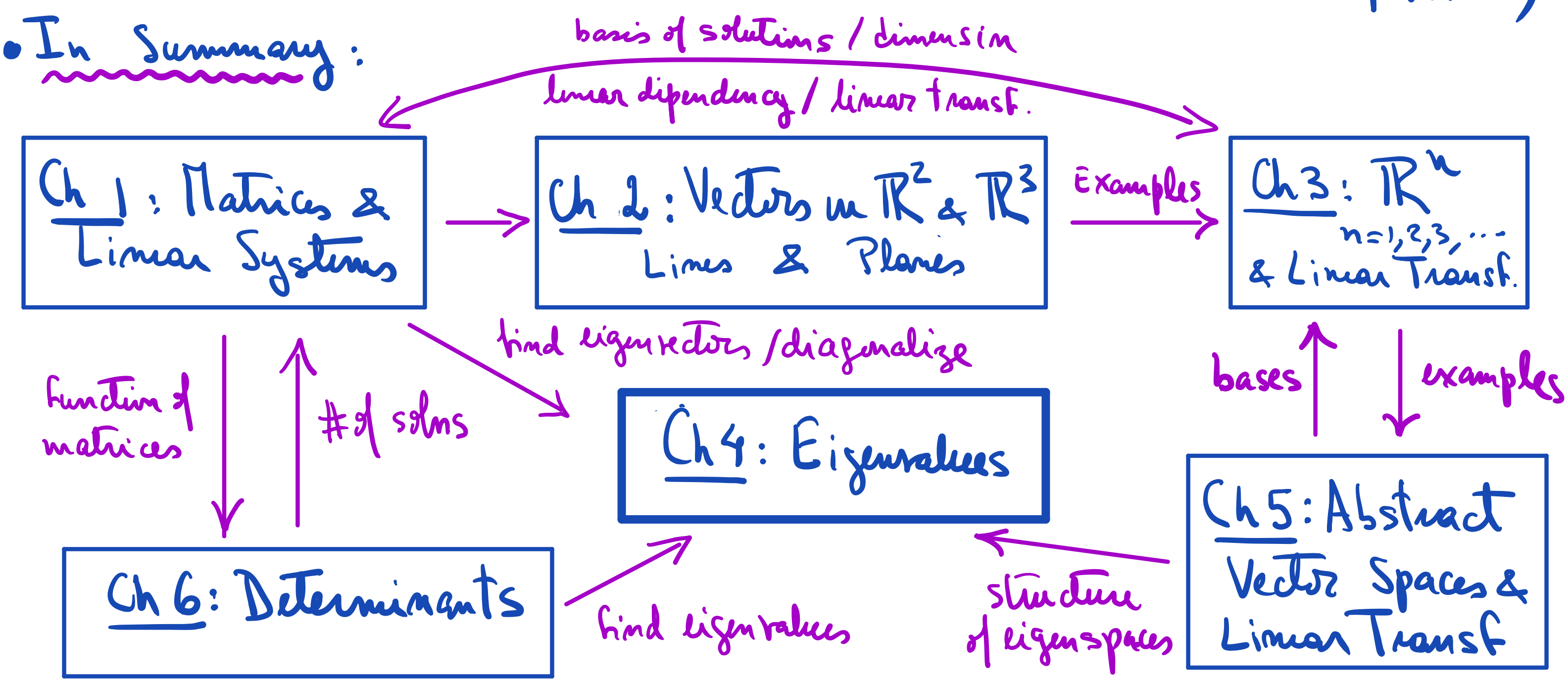
Lecture I: §1.1 Matrices and linear systems of equations

§1. Introduction:

- The main object of study in Linear Algebra is a system of linear equations. Such systems are conveniently encoded using matrices and are explicitly solved using an algorithm (Gauss-Jordan elimination)
- In this course, we will go over this method in detail; this lecture and the next one are devoted to this. For the main part of this course, we will see a re-interpretation of matrices, by viewing them as linear transformations between vector spaces
- The problem of solving a system of linear equations will also be rephrased in the language of vectors and linear dependence.
- Applications (statistical mechanics, computer networks, Markov processes) often require to solve the eigenvalue problem, which in turn combines **all** the concepts discussed earlier

  (Idea: Given a matrix A of size $n \times n$ & $N \gg 0$ large, compute A^N or its asymptotic behavior $\rightsquigarrow$ compute (largest) eigenvalues / diagonalize A if possible)

- In Summary:



(We will go over definitions and examples of all underlined words in the previous paragraphs – don't worry!)

Three objectives:

1. Solve the simplest types of equations in 2 or more variables
   (LINEAR!) (MATH 2153 - CALC III)

EXAMPLE: $X + 2Y + Z = 10$ describes a plane in $\mathbb{R}^3$ with normal direction $\langle 1, 2, 1\rangle$, passing through $(10, 0, 0)$.

2. Introduce underlying algebraic structures of these solution sets
   $\rightsquigarrow$ Vector spaces of finite dimension.

EXAMPLE: Solutions to homogeneous differential equations, eg $y'' + y = 0$

- $y(x) = a\boxed{\cos x} + b\boxed{\sin x}$ for any a, b real numbers
  basis for the space of solutions
- Can add two solutions & get another solution
- Can multiply a solution by a fixed number and get a soln.

"Plane" of Solns.

3. Applications: Solve the eigenvalue / eigenvector problem.

§1. Linear equations: (fix $n = 1, 2, 3, \ldots$)

Definition: A linear equation in n variables is an equation of the form

$$\boxed{a_1 X_1 + a_2 X_2 + \cdots + a_n X_n = b} \qquad (*)$$

- $a_1, a_2, \ldots, a_n, b$ are fixed numbers (usually real numbers, or complex numbers)
- $X_1, X_2, \ldots, X_n$ are unknowns (variables)

The word "linear" refers to the fact that the degree of each variable on the left-hand side of (*) is 1.

Examples: (1) $2x_1 + 3x_2 = 6$ is a linear equation in two variables. $\boxed{\begin{aligned} x_1 &= 3 \\ x_2 &= 0 \end{aligned}}$ is a solution; $\boxed{\begin{aligned} x_1 &= 0 \\ x_2 &= 2 \end{aligned}}$ is also a solution

(2) $x + 2y + z = 9$ is a linear equation in three variables x, y and z.

(For $n=2$, we are used to calling the variables x and y instead of x_1 and x_2. Similarly, for $n=3$ we are used to x, y, z, instead of x_1, x_2, x_3).

(3) $\sin(x_1) + \cos(x_2) = 1$ is NOT a linear equation.

Definition: A system of linear equations (or just a linear system) is a finite collection of linear equations.

Examples (1) $\begin{cases} x + y = 3 \\ 2x - y = 3 \end{cases}$ linear system of 2 equations in 2 variables (square system)

(2) $\begin{cases} x + y + 2z = 4 \\ 2x + y \quad\; = 0 \end{cases}$ linear system of 2 equations in 3 variables

Goal: Find solutions to a linear system of equations

Idea: Eliminate variables to get an equation with only one variable on its left-hand side.

Example (1) $\begin{cases} x+y = 3 \\ 2x-y = 3 \end{cases}$

Adding the two equations gives $3x = 6 \implies \boxed{x=2}$

Plug it back in the original equation to get $\boxed{y=1}$

Conclusion: The system has a unique solution : $\boxed{\begin{matrix} x=2 \\ y=1 \end{matrix}}$

(2) $\begin{cases} x+y+2z = 4 \\ 2x+y = 0 \end{cases} \leadsto \boxed{y=-2x}$

Replace y by $-2x$ in the first equation to get

$$-x+2z = 4 \implies x = 2z-4$$
$$y = -2x = -4z+8$$

So for any value of $z=t$ in $\mathbb{R}$ (real numbers), we have a solution $\boxed{\begin{matrix} x = 2t-4 \\ y = -4t+8 \\ z = t \end{matrix}}$. We see in this example that there are infinitely many solutions to the system.

(3) $\begin{matrix} x+3y = 4 \\ 2x+6y = 10 \end{matrix} \xrightarrow[\text{the 1st eqn by 2}]{\text{Multiply}} \begin{cases} 2x+6y = 8 \\ 2x+6y = 10 \end{cases}$

Substracting, we get $0=2$! We conclude the system has no solution at all.

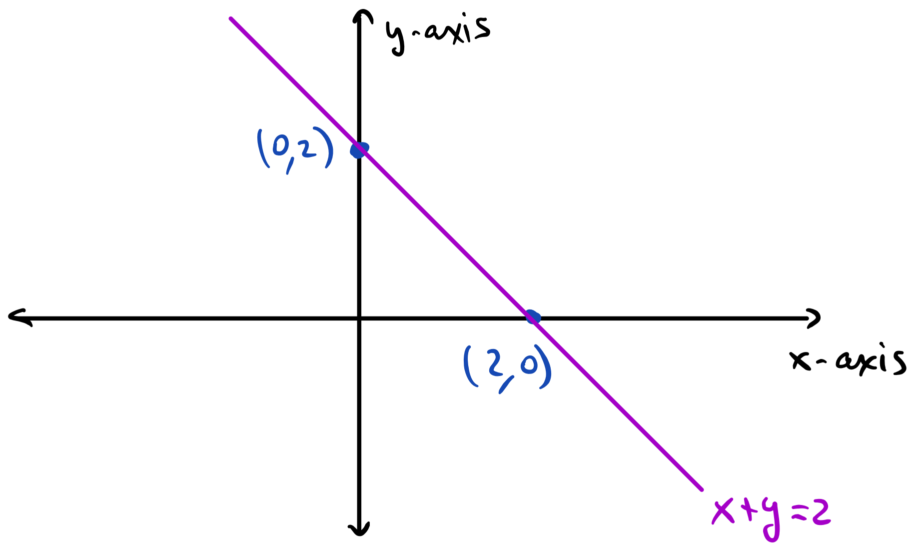
Observation: Later in the course, we will see that

> Every linear system of equations has either
> 0, 1 or ∞ number of solutions
> (∞ = infinity)

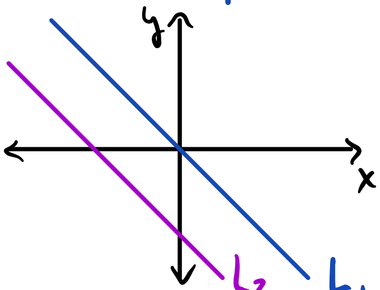
For $n = 2$ or 3 (number of variables is 2 or 3) we can see geometrically that this is true:

- **$n=2$:** Every linear equation in 2 variables defines a line in the plane $\mathbb{R}^2$.

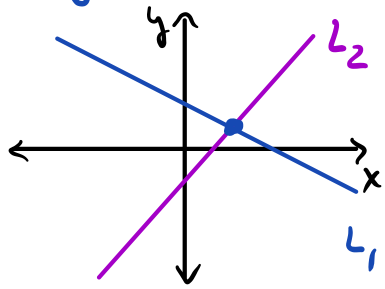
E.g. $x+y=2$



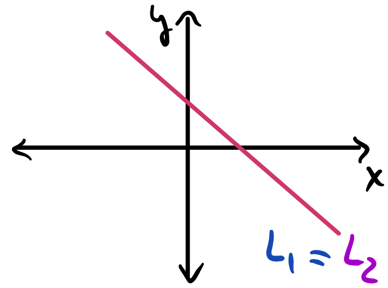
Assume we are given 2 linear equations in 2 variables. Geometrically, the two lines L_1 and L_2 defined by these equations are either parallel, or they intersect at a unique point, or $L_1 = L_2$



No Solution



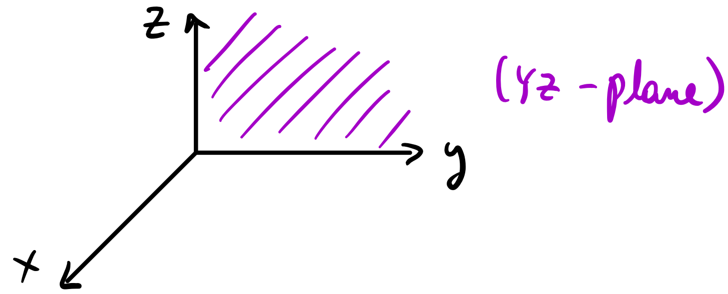
Unique Solution



∞-many Solutions

- **n = 3**: Every linear system in 3 variables defines a **plane** in $\mathbb{R}^3$

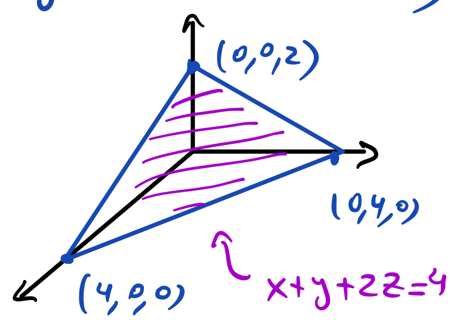
E.g. $x = 0$



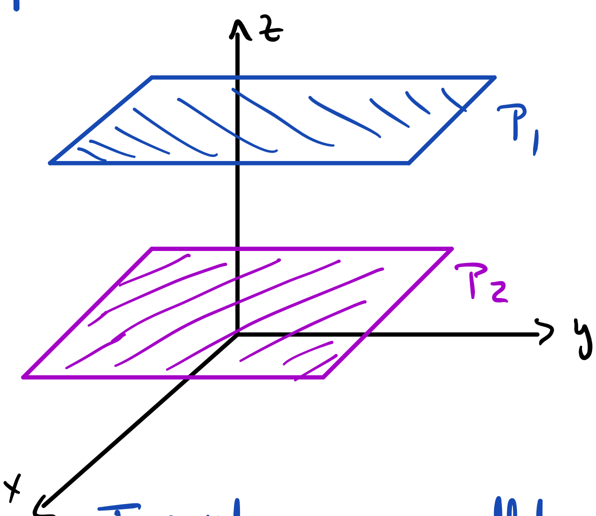
- If the linear system consists of **only one equation**, then there are infinitely many solutions (as many as the points in the plane it defines)

E.g. $x + y + 2z = 4$ For any choice of $x = s$, $y = t$ (s, t are real numbers) we have a solution

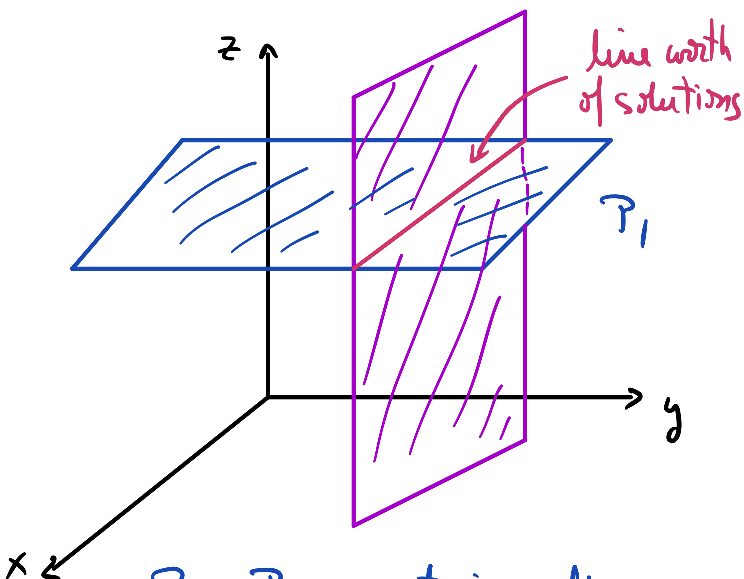
$$\boxed{\begin{aligned} x &= s \\ y &= t \\ z &= \frac{4-s-t}{2} \end{aligned}}$$



- If the linear system consists of 2 equations, then there are 2 possibilities:



Two planes are parallel (No solution)



P_1 & P_2 meet in a line (again, ∞ number of solutions)

E.g. System (2) on page 4 : $\begin{cases} x+y+2z=4 \\ 2x+y=0 \end{cases}$

Solutions of this system can be identified with the following line :

$$\begin{cases} x = 2t-4 \\ y = -4t+8 \\ z = t \end{cases}$$

⟵ parametric form of the line through $(-4,8,0)$ → set $t=0$, along the direction vector $\langle 2,-4,1\rangle$ ⇝ coeffs of t

Remarks : (1) Later in the course, we will see a proof of the fact that if #equations < # variables, then there are either no solutions, or infinitely many solutions.

(2) When a system admits infinitely many solutions, the set of solutions can be parameterized by a certain number of free parameters. We will also discuss the importance of this number in what will be called "rank-nullity theorem"

(eg above : $x+y+2z=4$ can be described using 2 free param

$\begin{cases} x+y+2z=4 \\ 2x+y=0 \end{cases}$ ——— 1 free param)