Lecture II: §1.1 Matrices and linear systems of equations II
§1.2 Echelon forms & Gauss–Jordan Elimination

Last time we saw an overview of the course, we introduced linear equations & set ourselves the goal of solving a system of linear equations.

Main result: Systems of linear equations have either none, unique or infinitely many solutions.

TODAY, we will formulate this goal precisely and go over a general method to simplify and solve a linear system.

Key idea:

MATRICES

$$\boxed{\begin{array}{c}\text{System of } m \text{ Linear equations}\\ \text{in } n \text{ unknowns}\end{array}} \rightsquigarrow \boxed{\begin{array}{c}\text{Array of numbers}\\ (mn+m \text{ many})\end{array}}$$

$$(*)\begin{cases} \text{Eqn } 1: & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \text{Eqn } 2: & a_{21}x_1 + a_{22}x_2 + \qquad + a_{2n}x_n = b_2 \\ \vdots & \vdots \\ \text{Eqn } m: & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \rightsquigarrow \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

- $a_{11}, a_{12}, \ldots, a_{1n}, \ldots, a_{m1}, \ldots, a_{mn}$ are fixed numbers (mn of them), called the coefficients of the linear system
- $b_1, b_2, \ldots, b_m$ are fixed numbers, called the constant terms.
- $x_1, x_2, \ldots, x_n$ are unknowns (variables)

§1 Matrices:

Definition: An $m \times n$ matrix is a rectangular array of numbers abbreviated as $A = (a_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$

i = row index
j = column index

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

\# of rows $= m$

\# of columns $= n$

Examples: (1) $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 9 \end{bmatrix}$ is a 2×3 matrix

(2) $\begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}$ is a 2×2 matrix (square matrix)

- If $m=n$, we say that A is a square matrix

§2. Representing a linear system as a matrix

A general linear system of m equations in n unknowns (*) as in page 1 is usually written as the following matrix of m rows & $n+1$ columns

$$B = [A|b] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

called the augmented matrix of the system (*). The matrix A is called the coefficient matrix of the system (*)

Example: Consider the following linear system of 3 equations in 3 unknowns

$$\begin{cases} x_1 - 2x_2 + x_3 = 4 \\ x_1 \qquad\quad + x_3 = 1 \\ \qquad x_2 - 2x_3 = 5 \end{cases} \rightsquigarrow A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \text{ coefficient matrix}$$

(Missing variables means coefficient $=0$)

(Good practice: align variables in the eqns)

$$B = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 5 \end{array}\right] \text{ augmented matrix}$$

§3. Elementary operations:

There are 3 operations we can perform on a linear system (augmented matrix) without changing its set of solutions. Write the equations as $E1, E2, \dots, Em$.

GOAL: $\boxed{\text{Simpler system} = \text{fewer terms}}$ $\longleftrightarrow$ $\boxed{\text{Simpler matrix } B \text{ (ECHELON FORM)}}$

- **Elementary Operation I**: SWAP. We can interchange 2 equations (if we swap E_i and E_j, we write it as $\boxed{E_i \longleftrightarrow E_j}$)

- **Elementary Operation II**: SCALE. We can choose a non-zero number say α, and replace E_i by αE_i, i.e. $\boxed{E_i \longrightarrow \alpha E_i}$.

- **Elementary Operation III**: COMBINE. We can replace E_i by $E_i + \lambda E_j$, where $j \neq i$ and λ is an arbitrary number, ie $\boxed{E_i \longrightarrow E_i + \lambda E_j}$

Theorem 1: Elementary operations do not change the set of solutions of the input linear system. (We get "equivalent systems")

Proof: It is clear that operations I and II can be reversed. Let us show why III can be reversed:

$$\boxed{\begin{array}{c}\text{System } S\\ E1\\ E2\\ \vdots\\ Em\end{array}} \xrightarrow[E_i \to E_i + \lambda E_j]{} \boxed{\begin{array}{c}\text{System } S'\\ E_i' = E_i + \lambda E_j\\ \text{rest unchanged}\end{array}} \xrightarrow[E_i' \to E_i' + (-\lambda) E_j']{} \boxed{\begin{array}{c}\text{Back to}\\ \text{system } S\end{array}}$$

$\square$

Observation: These operations can be performed on the ROWS of the augmented matrix producing simpler systems (with many 0's in staircase form) that are very easy to solve. This is the core of the Gauss–Jordan Elimination algorithm.

Example:

$$\begin{aligned} E_1 &: x_1 - x_2 + x_3 = 1 \\ E_2 &: x_1 + x_2 - x_3 = 5 \\ E_3 &: x_1 + 2x_2 + 4x_3 = 10 \end{aligned} \xrightarrow[\substack{E_2 \to E_2 - E_1 \\ E_3 \to E_3 - E_1}]{\text{Combine}} \begin{aligned} x_1 - x_2 + x_3 &= 1 \\ 2x_2 - 2x_3 &= 4 \\ 3x_2 + 3x_3 &= 9 \end{aligned}$$

$$\xrightarrow[\substack{E_2 \to \frac{1}{2}E_2 \\ E_3 \to \frac{1}{3}E_3}]{\text{Scale}} \begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ x_2 + x_3 &= 9 \end{aligned} \xrightarrow[E_3 \to E_3 - E_2]{\text{Combine}} \begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ 2x_3 &= 1 \end{aligned}$$

$$\xrightarrow[E_3 \to \frac{1}{2}E_3]{\text{Scale}} \begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ x_3 &= 1/2 \end{aligned}$$

Conclude
$$x_3 = \frac{1}{2}$$
$$x_2 = 2 + x_3 = 2 + \frac{1}{2} = 5/2$$
$$x_1 = 1 + x_2 - x_3 = 1 + \frac{5}{2} - \frac{1}{2} = 3$$

Check (in the original system) $x_1 - x_2 + x_3 = 3 - \frac{5}{2} + \frac{1}{2} = \frac{6-5+1}{2} = 1$ ✓

$$x_1 + x_2 - x_3 = 3 + \frac{5}{2} - \frac{1}{2} = \frac{6+5-1}{2} = 5 \checkmark$$
$$x_1 + 2x_2 + 4x_3 = 3 + 2\left(\frac{5}{2}\right) + 4\left(\frac{1}{2}\right) = 3 + 5 + 2 = 10 \checkmark$$

Operations on the matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 5 \\ 1 & 2 & 4 & 10 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1}]{} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & -2 & 4 \\ 0 & 3 & 3 & 9 \end{array}\right] \xrightarrow[\substack{R_2 \to \frac{1}{2}R_2 \\ R_3 \to \frac{1}{3}R_3}]{} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$\underset{R_3 \to R_3 - R_2}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

ECHELON / STAIRCASE FORM

§ 4 Echelon form:

Using 3 Elementary Row Operations (Swap, Scale, Combine) we can bring the augmented matrix of a system into Echelon form.

Definition: An $r \times s$ matrix B is said the be in Echelon form if

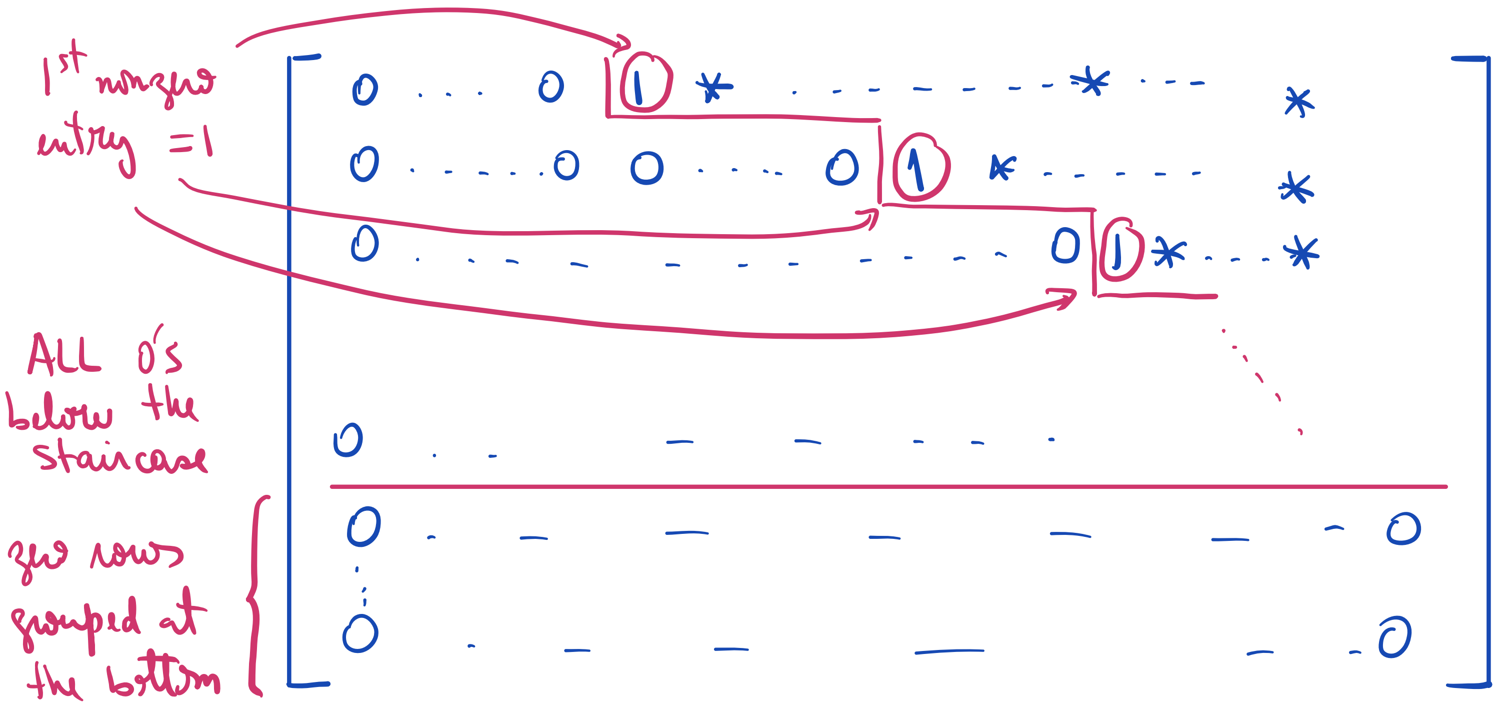
(i) all rows containing only zeroes are at the bottom of B

(ii) in every non-zero row, the first (from the left) non-zero entry is 1

(iii) if a row is nonzero, its first non-zero entry is to the right of the first non-zero entry of the previous row.

Intuitively, this means that the matrix has a "staircase shape"

(echelon = From the French word échelle, meaning ladder)



Definition: Reduced echelon form = echelon form + 0's above each starting 1.

Examples: (1) $\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ & $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are in echelon form
Last one in red echelon f.

(2) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ is not in echelon form, but it can be brought into it by elementary row operations.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{\text{Swap}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow[R_3 \to \frac{1}{4}R_3]{\text{Scale}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

Q Why do we care about echelon form matrices?

A: If the augmented matrix of a linear system is in echelon form, then the system is very easy to solve! We just need to solve from bottom to top. Furthermore, if solutions exist the number of free parameters needed to write all the solutions is

$$\boxed{\#\text{columns of } A - \#\text{ nonzero rows of } B}$$

- Solutions will exist if, and only if, # zero rows of A = # zero rows of B

- The elimination algorithm (Gauss-Jordan) produces a matrix in echelon form starting from any matrix via elementary row operations (Next Time)

Example: 3 equations in 4 variables
$$\begin{cases} x_2 + x_3 - x_4 = 3 \\ x_1 + 2x_2 - x_3 + x_4 = 2 \\ -x_1 + x_2 + 7x_3 - x_4 = 1 \end{cases}$$

$\rightsquigarrow$ Matrix $B = \left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 3 \\ 1 & 2 & -1 & 1 & 2 \\ -1 & 1 & 7 & -1 & 1 \end{array}\right]$ NOT in echelon form.

$\underset{\substack{\text{Swap} \\ R_1 \leftrightarrow R_2}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 3 \\ -1 & 1 & 7 & -1 & 1 \end{array}\right] \underset{\substack{\text{Combine} \\ R_3 \to R_3 + R_1}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 3 & 6 & 0 & 3 \end{array}\right]$

$\underset{\substack{\text{Combine} \\ R_3 \to R_3 - 3R_2}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 3 & 3 & -6 \end{array}\right] \underset{\substack{\text{Scale} \\ R_3 \to \frac{1}{3} R_3}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 1 & 1 & -2 \end{array}\right]$

Echelon form

The last matrix represent the system

$$\begin{cases} x_1 + 2x_2 - x_3 + x_4 = 2 \\ \qquad x_2 + x_3 - x_4 = 3 \\ \qquad\qquad x_3 + x_4 = -2 \end{cases}$$

zero rows of A $= 0$

—— B $= 0$

We solve from bottom to top using x_4 as the free parameter (# parameters $= 4 - 3 = 1$) Indeed, for any choice of $x_4 = t$ a real number, we get a solution:

$x_3 = -2 - t$; $x_2 = 3 - x_3 + x_4 = 3 - (-2-t) + t = 5 + 2t$

$x_1 = 2 - 2x_2 + x_3 - x_4 = 2 - 2(5+2t) + (-2-t) - t = -10 - 6t$

so we have infinitely many solutions

$$\boxed{\begin{aligned} x_1 &= -10 - 6t \\ x_2 &= 5 + 2t \\ x_3 &= -2 - t \\ x_4 &= t \end{aligned} \quad \text{for } t \in \mathbb{R}}$$

GAUSS-JORDAN ELIMINATION:

Input: Linear System
Output: Solution Set

- Step 1: Write the augmented matrix B of the input
- Step 2: Use Elementary Row operations to go from B to a matrix B' in echelon form.
- Step 3: Solve the system associated to B' from bottom to top.