Lecture III: §1.2 Gauss-Jordan Elimination

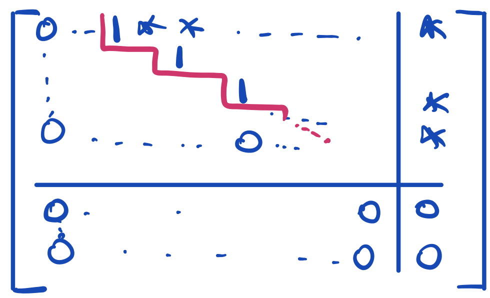
Recall: Last time we learned how to encode a linear system into a matrix $(A|b)$ called its <u>augmented matrix</u>. We listed <u>three elementary operations</u> which do not change the set of solutions of a linear system

	LINEAR SYSTEM $\longleftrightarrow$	AUGMENTED MATRIX
I SWAP	$E_i \longleftrightarrow E_j$	$R_i \longleftrightarrow R_j$
II SCALE	$E_i \longrightarrow \alpha E_i \quad (\alpha \neq 0)$	$R_i \longrightarrow \alpha R_i$
III COMBINE	$E_i \longrightarrow E_i + \lambda E_j \quad (i \neq j)$	$R_i \longrightarrow R_i + \lambda R_j$

Equations: $E_1, \ldots, E_m$ Rows: $R_1, \ldots, R_m$

- Ideal system to solve: those in <u>echelon form</u>

  We solve these from bottom to



<u>Examples</u>: $\begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 1 \end{bmatrix}$ reduced echelon form ; $\begin{bmatrix} 1 & 5 & 0 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ also in REF

$B = \left[\begin{array}{ccccc|c} 0 & 1 & 4 & 2 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ echelon form, <u>non reduced</u> $\longleftrightarrow \begin{cases} x_1 + 4x_2 + 2x_3 + 3x_4 = 1 \\ x_4 = 1 \\ 0 = 0 \end{cases}$

$\xrightarrow[R_1 \to R_1 - 3R_2]{\text{Combine}} \left[\begin{array}{ccccc|c} 0 & 1 & 4 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{cases} x_1 + 4x_2 + 2x_3 = -2 \\ x_4 = 1 \end{cases}$ $\rightsquigarrow$ $x_1 = -2 - 4x_2 + 2x_3$, $x_4 = 1$

$\rightsquigarrow$ <u>Solutions</u>: $\boxed{x_1 = -2 - 4s + 2t \quad x_2 = s \; ; \; x_3 = t \quad x_4 = 1 \qquad s, t \text{ real numbers}}$

- <u>Strategy</u>: simplify the input system by elementary operations by turning it into one in (reduced) echelon form.
- <u>TODAY</u>: we will see that this strategy always works, i.e., <u>any</u> matrix can be put in (reduced) echelon form using the 3 elementary operations.

Algorithm:

System $\rightsquigarrow$ $B = [A|b]$

$\updownarrow$ SAME SOLUTIONS $\qquad\downarrow$ Row Operations (*)

New System easy to solve $\leftsquigarrow$ $B' = [A'|b']$ with B' in REF

- dependent vars = # nonzero rows of A
- Free parameters = rest

§1 Gauss-Jordan Elimination:

Main Theorem Any matrix B can be put in reduced echelon form using the 3 elementary row operations. We find the output matrix using Gauss-Jordan Elimination Algorithm (Matrix B $\longrightarrow$ Echelon Form $\mathcal{E}(B)$ $\longrightarrow$ Reduced EF $\mathcal{RE}(B)$)

Sketch of Gauss-Jordan Algorithm:

- $B \to \mathcal{E}(B)$: Move from top-left to bottom-right to get the staircase shape, rescaling to start each nonzero row with ones. Swap to put all 0's rows at the bottom.
- $\mathcal{E}(B) \to \mathcal{RE}(B)$: Fix nonzero entries above each starting 1 via combine operation. Repeat columns from right to left.

Examples (1)

$$B = \begin{array}{c} \text{pivot} \\ \begin{bmatrix} 1 & 3 \\ 4 & 10 \end{bmatrix} \end{array} \xrightarrow[R_2 \to R_2 - 4R_1]{\text{Comb}} \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \xrightarrow[R_2 \to \frac{R_2}{-2}]{\text{Scale}} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \mathcal{E}(B)$$

$$\mathcal{RE}(B) = ? \quad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_1 \to R_2 - 3R_1]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{RE}(B)$$

(2)

$$B = \begin{bmatrix} 1 & 1 & -1 & 1 \\ \boxed{2} & -1 & 7 & 8 \\ \boxed{-1} & 1 & -5 & -5 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - \boxed{2}R_1 \\ R_3 \to R_3 - \boxed{(-1)}R_1}]{\text{Combine}} \begin{bmatrix} \boxed{1} & 1 & -1 & 1 \\ 0 & -3 & 9 & 6 \\ 0 & 2 & -6 & -4 \end{bmatrix}$$

Turn -3 into a 1 by scaling

next: fix this part

$$\xrightarrow[R_2 \to \frac{1}{-3} R_2]{\text{Scale}} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & \boxed{2} & -6 & -4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - \boxed{2}R_2]{\text{Combine}} \begin{bmatrix} 1 & \boxed{1} & -1 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathcal{E}(B)$$

$$\xrightarrow[R_1 \to R_1 - R_2]{\text{Combine}} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathcal{RE}(B)$$

GAUSS-JORDAN ALGORITHM:

Input : $B = [A|b]$ — m rows, $n+1$ columns

Output : $B' = [A'|b']$

STEP 0: If B has all zero entries, then $B' = B$

STEP 1: [Find Column] Pick the first (left-most) column with a nonzero entry (call it j)

STEP 2: [Swap Rows] Exchange Rows so that the j^{th} Column has $a_{1j} \neq 0$ (nonzero entry on row 1)

STEP 3: [Rescale Step] If $\alpha = a_{1j}$, then do $R_1 \longrightarrow \frac{1}{\alpha} R_1$ (after this, the new value of $a_{1j} = 1$)

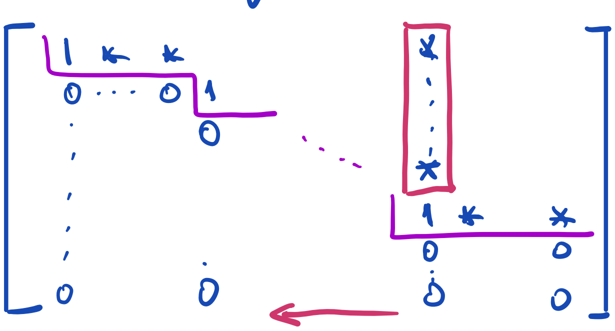
STEP 4: [Combine Step] Replace each nonzero row R_i for $i > 1$ with $R_i \to R_i - a_{ij} R_1$

$$\begin{array}{ccccccc} & & & j & & & \\ \end{array} \begin{bmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * \\ 0 & \cdots & 0 & a_{2j} & & & \\ \vdots & & \vdots & \vdots & & & \\ 0 & \cdots & 0 & a_{mj} & * & \cdots & * \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & * & \cdots & * \\ \vdots & & & \vdots & & B'' & \\ 0 & \cdots & 0 & 0 & * & \cdots & * \end{bmatrix}$$

B'' smaller matrix to fix

STEP 5: [Iterative Step] Run steps 0 to 4 for the smaller matrix B'' until we get an echelon form matrix. Call it $\mathcal{E}(B)$.

STEP 6: [From EF to REF] Work our way backwards to put 0's on top of each 1 starting a nonzero row of $\mathcal{E}(B)$ (using Combine Operation), fixing columns from right to left. The resulting matrix is $B' = \mathcal{R}\mathcal{E}(B)$.



Fix THESE FIRST and then move to the location of the 1 starting the previous row

§2. Solving REF systems:

Next GOALS: (1) Determine when a system has:

- no solution ("inconsistent")
- unique solution } ("consistent")
- infinitely many solutions }

(2) If ∞-many solutions, write the solutions in parametric form.

Observations: (1) If the matrix $\mathcal{RE}(B)$ contains the following row

$$[0 \; \cdots \; 0 \; 1]$$

then, the corresponding system has <u>no</u> solutions

(Why? Such row corresponds to the eqn: $0 \cdot x_1 + \cdots + 0 \cdot x_n = 1$, so $0=1$)

(2) The variables corresponding to columns of A' containing pivoted 1's (ie the corners of our staircase) are called <u>dependent variables</u>.

(3) The variables that are not dependent are called <u>independent</u> (or <u>free parameters</u>)

Examples (1) $B = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & -1 & 7 & 8 \\ -1 & 1 & -5 & -5 \end{array}\right] \rightsquigarrow \mathcal{RE}(B) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

pivoted 1's

- no row $[0 \; 0 \; 0 \; 1]$, so the system is consistent!
- Dependent Variables: x_1 & x_2, Free parameter: x_3

The system represented by $\mathcal{RE}(B)$ is $\begin{cases} x_1 + 2x_3 = 3 \\ x_2 - 3x_3 = -2 \end{cases}$

$x_3 = t$ free parameter $\rightsquigarrow \boxed{\begin{aligned} x_1 &= 3-2t \\ x_2 &= -2+3t \\ x_3 &= t \end{aligned}}$ are all the solutions to the system represented by B.

(2) $\begin{cases} x_1 + x_2 - x_5 = 1 \\ x_2 + 2x_3 + x_4 + 3x_5 = 1 \\ x_1 - x_3 + x_4 + x_5 = 0 \end{cases}$

Determine whether the system is consistent or not. If yes, describe its solutions in parametric form

$$B = \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 1 & 0 & -1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{\text{Combine}} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & -1 & -1 & 1 & 2 & -1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + R_2]{\text{Combine}} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right] = \mathcal{E}(B) \xrightarrow[R_2 \to R_2 - 2R_3]{\text{Combine}} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -3 & -7 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - R_2]{\text{Combine}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 6 & 0 \\ 0 & 1 & 0 & -3 & -7 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right] = \mathcal{RE}(B)$$

Conclusion: The system is consistent ($[0\ 0\ 0\ 0\ 0\ 1]$ is NOT a row of $\mathcal{RE}(B)$)

- Free parameters: x_4 & x_5. $\rightsquigarrow$ infinitely many solutions!

For every choice $x_4 = s$, $x_5 = t$ (2 numbers), we have a solution

$$\boxed{\begin{aligned} x_1 &= -3s - 6t \\ x_2 &= 3s + 7t + 1 \\ x_3 &= -2s - 5t \\ x_4 &= s \\ x_5 &= t \end{aligned}}$$

parametric form of ALL solutions