Lecture IV: §1.3 Consistent systems of linear equations

Last time: • We discussed Gauss-Jordan Elimination (Row Reduction) Algorithm to solve linear systems $(B \rightsquigarrow \mathcal{E}(B) \rightsquigarrow \mathcal{RE}(B))$

$$B = [A|b] \xrightarrow[\text{Row Operations}]{\text{Elementary}} B' = [A'|b'] \quad \text{in REF}$$

Main Results • B' is unique. (we will see later why)

• The systems associated to B & B' have the same solutions, but the one in reduced echelon form (associated to B') is EASIER to solve.

ALGORITHM:

System with m equations in n variables $\longrightarrow$ Augmented Matrix $B=[A|b]$ $\xrightarrow[\text{GAUSS JORDAN}]{\text{This is the lengthy part}}$ REF matrix B' $\longrightarrow$ Reduced system m eqns n vars $\longrightarrow$ SOLNS

§1. Some terminology:

Definition: We say two linear systems are <u>equivalent</u> if they have <u>the same set of solutions</u>.

Definition: Two matrices B_1 & B_2 are said to be <u>row equivalent</u> if we can go from B_1 to B_2 by performing elementary row operations.

Recall (from Lecture II), that elementary row operations are reversible. Thus, for example, if $\mathcal{RE}(B_1) = \mathcal{RE}(B_2)$, then B_1 and B_2 are row equivalent. In fact, this is the *only* way in which B_1 & B_2 can be row equivalent.

$$B_1 \underset{\text{row operations}}{\longleftarrow\!\!-\!-\!\!\longrightarrow} \mathcal{RE}(B_1) \underset{\text{(LATER)}}{=} \mathcal{RE}(B_2) \underset{\text{row operations}}{\longleftarrow\!\!-\!-\!\!\longrightarrow} B_2$$

Example: $B_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ & $B_2 = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$

$$B_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 3R_1]{} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \xrightarrow[R_2 \to \frac{R_2}{-2}]{} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 2R_2]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix} \xrightarrow[R_2 \to R_2 + R_1]{} \begin{bmatrix} 1 & 3 \\ 0 & 8 \end{bmatrix} \xrightarrow[R_2 \to \frac{R_2}{8}]{} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 3R_2]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Conclusion: B_1 & B_2 are row equivalent.

§2. Solving systems:

- Linear systems are classified into 2 types:

(1) Inconsistent. (no solutions)

Test: $[0 \cdots 0 \ 1]$ is a row of $RE(B)$

(2) Consistent (1 or ∞-many solns)

- dependent variables $\leftrightarrow$ starting 1's in $RE(B)$
- independ variables : rest

- Infinitely many solutions = we have at least 1 independent var. + system is consistent
- Unique solution = ALL variables are dependent + system is consistent

Conclusion: We can decide how many solutions a system has by looking at the matrix of the equivalent reduced system

Examples: (1) Solve
$$\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 \\ x_1 - 2x_2 - 2x_3 = -2 \\ -x_1 + 16x_2 + 2x_3 = 16 \end{cases}$$

Soln: $B = \left[\begin{array}{ccc|c} 2 & 3 & -4 & 3 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 16 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{} \left[\begin{array}{ccc|c} 1 & -2 & -2 & -2 \\ 2 & 3 & -4 & 3 \\ -1 & 16 & 2 & 16 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1}]{}$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & -2 \\ 0 & 7 & 0 & 7 \\ 0 & 14 & 0 & 14 \end{array}\right] \xrightarrow[R_2 \to \frac{R_2}{7}]{} \left[\begin{array}{ccc|c} 1 & -2 & -2 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 14 & 0 & 14 \end{array}\right] \xrightarrow[R_3 \to R_3 - 14R_2]{} \left[\begin{array}{ccc|c} 1 & -2 & -2 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 + 2R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \checkmark$$

no $[0\ 0\ 0\ 1]$ row, so the system is <u>consistent</u>.

x_1, x_2 are dependent variables
x_3 is independent $\rightsquigarrow$ infinitely many solutions.

Reduced System $\begin{cases} x_1 \quad -2x_3 = 0 \\ \quad x_2 \quad = 1 \end{cases} \rightsquigarrow \begin{array}{l} x_1 = 2x_3 \\ x_2 = 1 \end{array}$

<u>Solutions</u>: $\begin{array}{l} x_1 = 2t \\ x_2 = 1 \\ x_3 = t \end{array}$ where t is <u>any</u> real number

<u>Later on</u>: $(x_1, x_2, x_3) = (2t, 1, t) = (0,1,0) + t(2,0,1)$
[Vector notation for the solutions]

(2) Solve $\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 \\ x_1 - 2x_2 - 2x_3 = -2 \\ -x_1 + 16x_2 + 2x_3 = 0 \end{cases}$ (coefficient matrix is the same as in EX (1))

<u>Soln</u> $B = \left[\begin{array}{ccc|c} 2 & 3 & -4 & 3 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 0 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{} \left[\begin{array}{ccc|c} 1 & -2 & -2 & -2 \\ 2 & 3 & -4 & 3 \\ -1 & 16 & 2 & 0 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1}]{}$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & -2 \\ 0 & 7 & 0 & 7 \\ 0 & 14 & 0 & -2 \end{array}\right] \xrightarrow[R_2 \to \frac{R_2}{7}]{} \left[\begin{array}{ccc|c} 1 & -2 & -2 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 14 & 0 & -2 \end{array}\right] \xrightarrow[R_3 \to R_3 - 14R_2]{} \left[\begin{array}{ccc|c} 1 & -2 & -2 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -16 \end{array}\right]$$

$$\xrightarrow[R_3 \to \frac{R_3}{-16}]{} \left[\begin{array}{ccc|c} 1 & -2 & -2 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] = B'$$

in Echelon Form

New system: $\begin{cases} x_1 - 2x_2 - 2x_3 = -2 \\ \quad x_2 \quad = 1 \\ \boxed{0 = 1} \end{cases}$

- $[0\ 0\ 0\ 1]$ is a row of B' so the original system is inconsistent because the reduced one is.

Observation: We perform the same row operations in both examples because the coefficient matrix is the same!

We can solve both systems at the same time, by working with

$$\left[\begin{array}{c|c|c} A & \begin{matrix} b \text{ for} \\ \text{system } 1\end{matrix} & \begin{matrix} b \text{ for} \\ \text{system } 2\end{matrix}\end{array}\right] = \left[\begin{array}{ccc|c|c} 2 & 3 & -4 & 3 & 3 \\ 1 & -2 & -2 & -2 & -2 \\ -1 & 16 & 2 & 16 & 0 \end{array}\right]$$

(3)
$$\left[\begin{array}{ccccc|c} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- consistent
- 3 dependent variables: x_2, x_4, x_5
- 2 independent variables: x_1, x_3,

So we have infinitely many solutions!

Solution: Row 3: $x_5 = 2$

Row 2: $x_4 = 3$

Row 1: $x_2 = x_3 = t$

x_1 is free $\quad x_1 = s$

t, s are real numbers (arbitrary)

Vector expression: $(x_1, x_2, x_3, x_4, x_5) = (s, t, t, 3, 2)$

$$= s(1,0,0,0,0) + t(0,1,1,0,0) + (0,0,0,3,2)$$

§3. Homogeneous systems:

Definition: A system
$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$
is said to be homogeneous if $b_1 = b_2 = \cdots = b_n = 0$.

Observation: Homogeneous systems are always consistent, since $x_1 = x_2 = \cdots = x_n = 0$ is certainly a solution. We call it the trivial solution.

So homogeneous systems have either 1 or ∞-many solutions.

Example:
$$\begin{cases} 2x_1 + 3x_2 - 4x_3 = 0 \\ x_1 - 2x_2 - 2x_3 = 0 \\ -x_1 + 16x_2 + 2x_3 = 0 \end{cases}$$

$$B = \left[\begin{array}{ccc|c} 2 & 3 & -4 & 0 \\ 1 & -2 & -2 & 0 \\ -1 & 16 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 2 & 3 & -4 & 0 \\ -1 & 16 & 2 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 14 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to \frac{R_2}{7}} \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 14 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - 14R_2} \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \checkmark$$

no $[0\ 0\ 0\ 1]$ row, so the system is consistent.

- x_1, x_2 dependent variables
- x_3 independent

Solutions: $\begin{cases} x_1 = 2x_3 \\ x_2 = 0 \end{cases}$ $\rightsquigarrow (x_1, x_2, x_3) = x_3(2, 0, 1)$

§4. Consequences of Gauss-Jordan:

① A linear system has 0, 1 or ∞-many solutions

② If # equations < # variables, the linear system cannot has a unique solution

(**Reason:** # dependent variables $\leq$ # eqns < # variables, so we will have at least one independent variable)