Lecture V: §1.3 Consequences of Gauss-Jordan
§1.5 Matrix Operations

§1. Consequences of Gauss-Jordan:

① A linear system has 0, 1 or ∞-many solutions

② If # equations < # variables, the linear system cannot has a unique solution
(Reason: # dependent variables $\leq$ # eqns < # variables, so we will have at least one independent variable)

Definition: Assume the matrix B' is in reduced echelon form. We define $\text{Rank}(B') = r = $ # nonzero rows of B' = # dependent vars.

For arbitrary matrices B $\text{Rank}(B) = \text{Rank}(B')$ where B' is the unique reduced echelon form matrix equivalent to B.

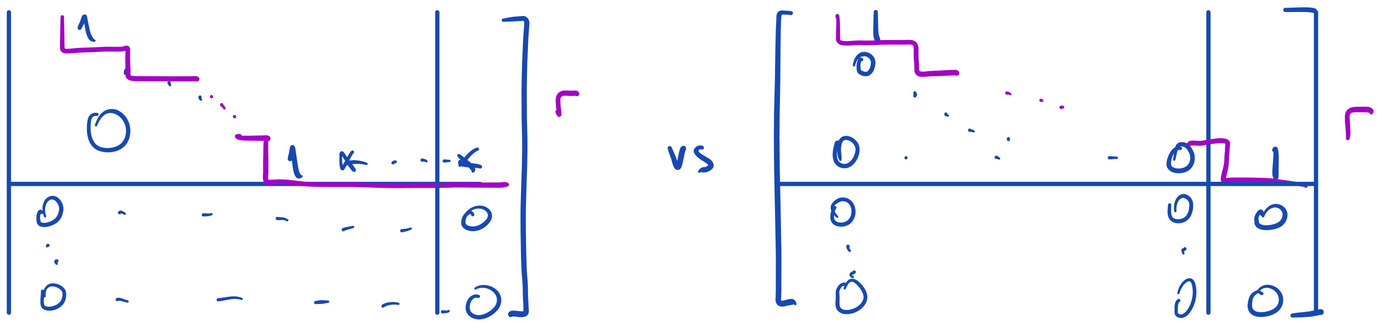
Consequences: ① $\text{Rank}(B') \leq m$ = # rows of B'
② $\text{Rank}(B') \leq n+1$ = # columns of B'
③ $\text{Rank}(B') \leq n$ if the system is consistent
④ # Free parameters $= n - \text{Rank}(B')$, so we have a unique solution if and only if the system is consistent & $\text{Rank}(B') = n$.
(note that $n \leq m$ by ①)

Why ③?

$$B' = \left[\begin{array}{ccc|c} 0 \ 1 & 1 \ \cdots & & \\ \hline 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{array}\right] \begin{array}{l} \} \; r = \text{Rank}(B') \\ \} \; \text{zero rows} \end{array}$$

$\underbrace{\qquad\qquad}_{n}$

If B' has $[0, \cdots 0, 1]$ as a row, we are inconsistent. This means by our assumptions that the right-most starting 1 is within the first n columns. The staircase shape forces $r \leq n$.



consistent | inconsistent

§2. Operations on Matrices:

Matrices = rectangular arrays of numbers

NEXT GOALS: ① Define 3 operations on matrices → Addition, Scalar Multiplication, Multiplication

(these 3 will be used to define abstract vector spaces)

② Study the algebra behind these operations (rules that will help us compute faster, like we do with $+$ & $\times$ in $\mathbb{R}$)

Application: We can write linear systems
$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$
using matrix multiplication
$$\underset{m\times \underline{\underline{n}}}{A}\underset{\underline{\underline{n}}\times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} = \underset{m\times 1}{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}} \qquad A = (a_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq m}}$$

Definition: Two matrices are equal when they have the same size & the same entries. More explicitly:

if $A = (a_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$ & $B = (b_{ij})_{\substack{1\leq i\leq r \\ 1\leq j\leq s}}$ are two matrices of sizes $m\times n$ & $r\times s$, respectively then $A = B$ means

$m = r$, $n = s$ and $a_{ij} = b_{ij}$ for every $i = 1, \ldots, m$ & $j = 1, \ldots, n$

Examples: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ because $(1,2)$ entries don't agree

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ because the number of columns is different.

Definition: A **scalar** is a number (real or complex)

- Next we define addition and scalar multiplication, and study their properties.

§3. Matrix Addition & Scalar Multiplication:

- Fix two matrices A & B of the same size. We want to build a new matrix $A+B$

Definition 1: If A & B have size $m \times n$, then $A+B$ is a matrix of size $m \times n$ with entries $(A+B)_{i,j} = A_{ij} + B_{ij}$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$

(add entry by entry)

Examples: (1) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1+3 & 0+1 \\ 0+2 & 1+0 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is not defined (the matrices have different sizes!)

- Given a matrix A and a scalar r we want to build a new matrix rA

Definition 2: If A has size $m \times n$, then the **scalar product** rA is a matrix of size $m \times n$ with entries $(rA)_{ij} = rA_{ij}$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$

(multiply all entries by r)

Example $2 \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 3 \\ 2 \cdot 0 & 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$

§4. Matrix multiplication:

- Next, we define multiplication of matrices. We will discuss the definition & do some examples. Next time, we will see why this is the "right definition".

Idea:

$$\boxed{\underset{m\times n}{A} \cdot \underset{n\times s}{B} \rightsquigarrow \underset{m\times s}{AB}}$$

must match, or else the product is not defined

Definition: Given $A = (a_{ij})$ $m\times n$ matrix, the product $A\cdot B$
$B = (b_{ij})$ $r\times s$ ——

is only defined when $n = r$ (# cols A = # rows B). Assuming this is the case, then $A\cdot B$ is a matrix of size $m\times s$ with entries

$$(A\cdot B)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$
$$= \sum_{k=1}^{n} a_{ik}b_{kj} \quad \text{(summation notation)}$$

Pictorially: To get $(i,j)^{th}$ entry of $A\cdot B$, we focus on the i^{th} row of A & the j^{th} column of B

$$[a_{i1}\ a_{i2}\ \cdots\ a_{in}] \qquad \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

. We multiply the corresponding entries & add them up:

$$\rightsquigarrow a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = (ij) \text{ entry of } AB$$

Examples: (1) $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is NOT defined

$\quad\quad 2\times \underline{\underline{3}} \quad\quad \underline{\underline{2}}\times 2$

(2) $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 7 \\ -2 & 4 & 5 \end{bmatrix}$ is defined and gives a 2×3 matrix

$\quad\quad 2\times \underline{\underline{3}} \quad\quad \underline{\underline{3}}\times 3$

- (1,1) entry : $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = 1\cdot 1 + 0\cdot 0 + 3\cdot(-2) = -5$

- (1,2) — : $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = 1\cdot(-1) + 0\cdot 2 + 3\cdot 4 = -1+12 = 11$

- (1,3) — : $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} = 1\cdot 3 + 0\cdot 7 + 3\cdot 5 = 3+15 = 18$

Similarly, we can compute the (2,1), (2,2) & (2,3) entries.

Result: $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 7 \\ -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -5 & 11 & 18 \\ -1 & 7 & 22 \end{bmatrix}$

Note: The formula for each entry of $A\cdot B$ is exactly how we define dot products in $\mathbb{R}^2$ & $\mathbb{R}^3$. This is more general.