Lecture VI: §1.6 Algebraic properties of matrix operations

Last time: we defined addition, scalar multiplication & product of matrices

Operation	Input	Output
① Addition	Two matrices $A=(a_{ij})$ and $B=(b_{ij})$ of the same size $(m\times n)$	$A+B$: an $m\times n$ matrix with $(A+B)_{ij}=A_{ij}+B_{ij}$
② Scalar Multiplication	λ : a number; $A=(a_{ij})$ an $m\times n$ matrix	$\lambda A=$ matrix of size $m\times n$ with $(\lambda A)_{ij}=\lambda A_{ij}$
③ Product	Two matrices $A=(a_{ij})$ $m\times n$, $B=(b_{ij})$ $n\times s$ (# cols(A) = # rows(B))	AB = matrix of size $m\times s$; m = # rows(A), s = # cols(B) (*)

(*) **Formula:** $(AB)_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}$

$$= [i^{th} \text{ row of } A] \begin{bmatrix} j^{th} \\ \text{column} \\ \text{of } B \end{bmatrix}$$

Warm up: A $m\times n$, $\underline{x}=n\times 1$ (column vector) $\rightsquigarrow A\cdot\underline{x}$ is a column vector of size m.

$$A\,\underline{x} = x_1\,col_1(A)+x_2\,col_2(A)+\cdots+x_n\,col_n(A)$$

General case: A $m\times n$, B $n\times s$ $\rightsquigarrow Col_j(AB)=A\cdot Col_j(B)$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}_{2\times 3}$ $\qquad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 4 & 1 & 10 \\ 0 & 5 & 1 & 0 \end{bmatrix}_{3\times 4}$

(1) BA is NOT defined (sizes are incompatible)

(2) AB is a 2×4 matrix $= \begin{bmatrix} -2 & 23 & 6 & 20 \\ -1 & -1 & 0 & 10 \end{bmatrix}$

$$\mathrm{Col}_1(AB) = A\,\mathrm{Col}_1(B) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$\mathrm{Col}_2(AB) = A\,\mathrm{Col}_2(B) = \begin{bmatrix} 4\cdot 2 + 3\cdot 5 \\ 4 - 5 \end{bmatrix} = \begin{bmatrix} 23 \\ -1 \end{bmatrix}$$

$$\mathrm{Col}_3(AB) = A\,\mathrm{Col}_3(B) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\mathrm{Col}_4(AB) = A\,\mathrm{Col}_4(B) = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

Q: Why this definition?

A1: Nice algebraic properties (next!)

A2: Allows for fast substitution (composition of linear functions)

Ex: Combine $\begin{cases} 1 = 3y_1 - y_2 + y_3 \\ 2 = -3y_1 + 5y_2 \end{cases}$ & $\begin{cases} y_1 = -4z_1 + z_3 \\ y_2 = z_2 - z_3 \\ y_3 = 0 \end{cases}$

into a single linear system in (z_1, z_2, z_3)

Use $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ & $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

To get $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -1 & 1 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{= \begin{bmatrix} -12 & -1 & 4 \\ 12 & 5 & -8 \end{bmatrix}} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$. Conclude that

$$\begin{cases} 1 = -12z_1 - z_2 + 4z_3 \\ 2 = 12z_1 + 5z_2 - 8z_3 \end{cases}$$

§2. Algebraic Properties

SLOGAN: Matrix operations are almost as nice as operations in $\mathbb{R}$

Theorem 1: A, B, C $m \times n$ matrices. Then:

1. [Commutative] $A+B = B+A$
2. [Associative] $(A+B)+C = A+(B+C)$
3. [Neutral Element] The zero matrix $\mathbb{O}$ of size $m \times n$ (all entries are 0) satisfies $A + \mathbb{O} = \mathbb{O} + A = \mathbb{O}$ for all matrices A of size $m \times n$
4. [Additive Inverse] Given A, the matrix P of size $m \times n$ with entries $P_{ij} = -A_{ij}$ for all i,j solve the matrix equation in P

$$A + P = P + A = \mathbb{O}.$$

Q Why is this true?

A: Addition for matrices is done entry-by-entry & these properties are true in $\mathbb{R}$. (= 1×1 matrices)

Obs: $\mathbb{O}$ is sometimes denoted by $\mathbb{O}_{m \times n}$ if the size is not clear.

$$\mathbb{O}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbb{O}_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Definition: The Identity Matrix of size $n \times n$ (denoted by I_n) is the square matrix with 1's in the diagonal and 0's elsewhere.

$$I_n = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}}_{n} \Big\} n$$

↳ diagonal: (i,i) entries

Ex $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Theorem 2: A of size $m\times n$, B of size $n\times s$ & C of size $s\times l$

① [Associative] $\underbrace{(AB)}_{m\times s}\underbrace{C}_{s\times l} = \underbrace{A}_{m\times n}\underbrace{(BC)}_{n\times l}$ $m\times l$ matrices

② [Associative II] α,β scalars $\alpha(\underbrace{\beta A}_{m\times n}) = (\alpha\beta)\underbrace{A}_{m\times n}$ $m\times n$ matrix

③ [Associative III] α scalar $\alpha(AB) = (\alpha A)B = A(\alpha B)$

④ [Neutral Element] $A = \underbrace{A}_{m\times n}\underbrace{I_n}_{n\times n} = \underbrace{I_m}_{m\times m}\underbrace{A}_{m\times n}$

Q Why? A Explicit computation of each entry, once sizes have been determined (see textbook)

Next: Relate the 3 operations!

Theorem 3:

① [Distribution I] Fix A, B of size $m\times n$, C of size $n\times s$. Then

$$\underbrace{(A+B)}_{m\times n}\underbrace{C}_{n\times s} = \underbrace{AC}_{m\times s} + \underbrace{BC}_{m\times s} \qquad m\times s \text{ both sides}$$

② [Distribution II] Fix A of size $m\times n$, B, C of size $n\times s$. Then

$$\underbrace{A}_{m\times n}\underbrace{(B+C)}_{m\times s} = \underbrace{AB}_{m\times s} + \underbrace{AC}_{m\times s} \qquad m\times s \text{ both sides}$$

③ [Distribution III] Fix α,β scalars, A of size $m\times n$. Then

$$(\alpha+\beta)A = \alpha A + \beta A \qquad m\times n \text{ both sides}$$

④ [Distribution IV] Fix α scalar, A, B of size $m\times n$. Then

$$\alpha\underbrace{(A+B)}_{m\times n} = \underbrace{\alpha A}_{m\times n} + \underbrace{\alpha B}_{m\times n} \qquad m\times n \text{ both sides.}$$