Lecture VII: §1.6 Algebraic properties of matrix operations
The Euclidean space $\mathbb{R}^n$.

Last Time: Discussed algebraic properties of addition, scalar mult. & product of matrices.

In particular, we have 2 Neutral Elements:

(1) For Addition: $\mathbb{O}$ = zero matrix $(m\times n)$ $\quad A = \mathbb{O} + A = A + \mathbb{O}$ $\quad A: m\times n$

(2) For Product: I = Identity matrix (square matrix!) $= \underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{s}$ s $\quad A = I_m A = A I_n$

TODAY:
- One more operation = transpose
- Euclidean space $\mathbb{R}^n$
- Invertible matrices (w.r.t. product)

§1. Transpose of a matrix:

IDEA: Transposing means swapping the role of rows & columns.

Definition: Given a matrix A of size $m\times n$, the transpose of A is a matrix A^T of size $n\times m$ with entries $(A^T)_{ij} = A_{ji}$ $\quad i=1,\ldots,n$; $j=1,\ldots,m$

Example $\quad A = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{bmatrix}}_{2\times 3} \rightsquigarrow A^T = \underbrace{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 0 \end{bmatrix}}_{3\times 2}$

Next, we show that this new operation interacts very well with the other 3.

Theorem: Fix A, B of size $m\times n$, C of size $n\times \ell$. Then

① $(A+B)^T = A^T + B^T$ $\qquad n\times m$ both sides

② $(A^T)^T = A$ $\qquad m\times n$ " "

③ $\underbrace{(AC)^T}_{\substack{m\times \ell \\ \ell\times m}} = \underbrace{C^T}_{\ell\times n} \underbrace{A^T}_{n\times m}$ $\qquad \ell\times m$ " "

Q Why? A ① & ② are easy to check. Let's discuss ③.

$$(AC)^T_{ij} = (AC)_{ji} = A_{j1}C_{1i} + A_{j2}C_{2i} + \cdots + A_{jn}C_{ni}$$

for each $i=1,\dots,l$, $j=1,\dots,m$

$$= C_{1i}A_{j1} + C_{2i}A_{j2} + \cdots + C_{ni}A_{jn}$$

$$= (C^T)_{i1}(A^T)_{1j} + (C^T)_{i2}(A^T)_{2j} + \cdots + (C^T)_{in}(A^T)_{nj}$$

$$= (C^T A^T)_{ij}.$$

Definition: We say a matrix A is symmetric if $A^T = A$

In particular A must be a square matrix $(m=n)$.

Why? Symmetric matrices are diagonalizable with real eigenvalues (later!)

Proposition: If A has size $m\times n$, then:

① AA^T is symmetric of size $m \times m$

② A^TA ______ $n\times n$

Why? $\underset{m\times n}{A}\,\underset{n\times m}{A^T}$ has size $m\times n$ & $\underset{m\times n}{A^T}\,\underset{n\times m}{A}$ has size $m \times$ m

$$(AA^T)^T \underset{\text{by Theorem}}{=} (A^T)^T A^T \underset{\text{by Theorem}}{=} AA^T \quad \& \quad (A^TA)^T = A^T(A^T)^T = A^TA$$

§2. Euclidean space $\mathbb{R}^n$ & Dot Product

- Addition & scalar multiplication match the usual operations for vectors in $\mathbb{R}^2, \mathbb{R}^3, \ldots$
- We will write solutions to linear systems of m equations & n unknowns as columns $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. We will denote the space of all such columns $\mathbb{R}^n$ and call it Euclidean space of dimension n

$$n\times 1 \text{ matrices} = \mathbb{R}^n = \left\{ \vec{v} : \vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ where } x_1, x_2, \ldots, x_n \text{ are real numbers} \right\}$$

- $\mathbb{R}^n$ has 2 operations : addition & scalar multiplication
- Extra operation in $\mathbb{R}^n$ = dot product
- Definition: given two vectors $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ & $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ we define

the dot product $\vec{v} \cdot \vec{u}$ as the product $[v_1, \ldots v_n] \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$

Using the transpose, we have $\vec{v} \cdot \vec{u} = \underset{1\times n}{\vec{v}^T} \; \underset{n\times 1}{\vec{u}}$.

- Definition The norm or magnitude of a vector $\vec{v}$ in $\mathbb{R}^n$ equals

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \cdots + v_n^2} = \sqrt{\vec{v}^T \vec{v}}$$

We call it the Euclidean length of $\vec{v}$.

Example $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

- $\vec{x}^T \vec{y} = [1\ 0\ {-1}] \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 1\cdot 2 + 0\cdot 1 + (-1)\cdot 3 = -1$

- $\vec{x}^T \vec{x} = [1\ 0\ {-1}] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1^2 + 0^2 + (-1)^2 = 1+1 = 2$

  so $\|\vec{x}\| = \sqrt{2}$

- $\vec{y}^T \vec{y} = [2\ 1\ 3] \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 2^2 + 1^2 + 3^2 = 14$

  so $\|\vec{y}\| = \sqrt{14}$

- $\|\vec{x} - \vec{y}\| = \left\| \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix} \right\| = \sqrt{[-1, -1, -4] \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}} = \sqrt{(-1)^2 + (-1)^2 + (-4)^2}$

  $= \sqrt{18}$.

• Advantage of $\mathbb{R}^n$ structure: We can write solutions to linear systems in vector form:

Example: $B = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right]$ REF

dependent

$$\begin{cases} x_1 - x_3 = 1 \\ x_2 + x_3 = 3 \end{cases}$$

x_1, x_2 dependent
x_3 independent

$(x_1, x_2, x_3) = (1+x_3, 3-x_3, x_3)$

We will write it as $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1+x_3 \\ 3-x_3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

matrix addition

particular solution

scalar product

Note: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ solves the homogeneous system $\begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$

This is a general phenomenon!

Proposition: If an $m \times n$ system with augmented matrix $B = [A|b]$ is consistent, then the general form of a solution is:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} + \text{General Solution of the homogeneous system with augmented matrix } [A|0]$$

$$\underset{\vec{x}}{\shortparallel} \qquad \underset{\vec{p}}{\shortparallel} \qquad \qquad \shortparallel$$

$$t_1 [\;\;] + \cdots + t_s [\;\;] \quad s = \#\text{ indep variables}$$

Why is this true? We know $A\vec{x} = \vec{b}$ & $A\vec{p} = \vec{b}$

Any vector $\vec{u} = \vec{x} - \vec{p}$ solves the associated homogeneous system.

since $A\vec{u} = A(\vec{x} - \vec{p}) = A\vec{x} - A\vec{p} = \vec{b} - \vec{b} = \vec{0}$ □

Example $\begin{cases} x_1 - x_2 + 2x_3 = 0 \\ x_4 - x_5 = 0 \end{cases}$ $\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$ REF

x_1, x_4 : dependent variables
x_2, x_3, x_5 : independent variables

$x_1 = x_2 + 2x_3$
$x_4 = x_5$ gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 + 2x_3 \\ x_2 \\ x_3 \\ x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

x_2, x_3, x_5 free

General form of the solution

§3 Multiplication of numbers vs. matrices

(1) $ab = ba$ for numbers **but** $AB \neq BA$ for matrices

Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\neq$ $BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(2) $ab = 0$ means either $a = 0$ or $b = 0$ for numbers **but**

$AB = 0$ can hold with $A \neq 0$ & $B \neq 0$. (Example above)

(3) $a \neq 0$ means we can always find $b = \frac{1}{a}$ with $ab = 1$

but there are nonzero matrices (A) for which $AB = I$ or $BA = I$ has no solution. (example above)

Example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Let us try to solve $AB = I_2$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \overset{B}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}}_{= \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

cannot be solved because of (2,2) entry ($0 \neq 1$) so no B can work!

Next Time: Invertible matrices!