Lecture VIII §1.9 Matrix Inverses

Recall • I_n = identity matrix of size $n\times n$ = $\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots\ 1 \end{bmatrix}$ $\left(I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$

• If A is an $m\times n$ matrix $\rightsquigarrow$ $AI_n = I_m A = A$ (Identity $\leftrightarrow$ 1 in $\mathbb{R}$)

§1 Multiplication of numbers vs. matrices

(1) $ab = ba$ for numbers but $AB \neq BA$ for matrices

Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\neq$ $BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(2) $ab = 0$ means either $a=0$ or $b=0$ for numbers but

$AB = \mathbb{0}$ can hold with $A \neq \mathbb{0}$ & $B \neq \mathbb{0}$. (Example above)

(3) $a \neq 0$ means we can always find $b = \frac{1}{a}$ with $ab = 1$

but there are nonzero matrices (A) for which $AB = I$ or $BA = I$ has no solution. (example above) I = Identity matrix

Example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Let us try to solve $AB = I_2$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \overset{B}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}}_{= \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

cannot be solved because of (2,2) entry ($0 \neq 1$)
so no B can work!

§2. Invertible matrices:

Definition: An $n\times n$ matrix A is invertible if there exists an $n\times n$ matrix B satisfying $AB = I_n = BA$.

Such a matrix B is called the "inverse of A". It is unique, and we denote it by A^{-1}.

Q Why is it unique? **A**: If B, B' are two $n\times n$ matrices with
$AB = BA = I_n$ & $AB' = B'A = I_2$, then

$$B = BI_n \underset{\text{use } I_n = AB'}{=} B(AB') \underset{\text{Assoc}}{=} (BA)B' \underset{\text{use } BA = I_n}{=} I_nB' = B'.$$

⚠ A^{-1} doesn't always exist (example above)

Example: (1) I_n is always invertible since $I_nI_n = I_n$
so $I_n^{-1} = I_n$

(2) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible with $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Check: $AA^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & -1+1 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$A^{-1}A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & -1+1 \\ 0+1 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

Significance of invertible matrices for linear systems.

Proposition: If $A \overset{= x}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} = \overset{= b}{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}$ is a system with n eqns & n unknowns
and A is invertible, then the system has a unique solution,
namely $A^{-1}\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

Why? • $A\left(A^{-1}\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}\right) = (AA^{-1})\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = I_n\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

So it is a solution.

• It is the unique solution: If, $A\underline{x} = \underline{b}$ multiply both sides by A^{-1} on the right:
$$A^{-1}(A\underline{x}) = A^{-1}\underline{b}$$

But $A^{-1}(A\underline{x}) \overset{\text{Assoc.}}{=} (A^{-1}A)\,\underline{x} = I_n\underline{x} = \underline{x}$

Conclude: $\underline{x} = A^{-1}\underline{b}$ is the only solution to the system.

Observation: This is the most important feature of invertible matrices, which in fact characterizes them, and allows us to compute A^{-1} using Gauss-Jordan Elimination. This will also show the converse of the assertion in Proposition 1, namely

Proposition 2: Fix a matrix A of size $n\times n$. If every linear system $A\vec{x}=\vec{b}$ has a unique solution, then A is invertible.

Q Why?

A Build n convenient linear systems by varying $\vec{b}$.
We know by our hypothesis that these systems have unique solutions.

Take $\vec{b}$ to be
$$\underbrace{\vec{e}_1 = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix},\ \vec{e}_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix},\ \ldots,\ \vec{e}_n = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}}_{\text{called coordinate vectors in } \mathbb{R}^n}$$

These are the columns of the identity matrix I_n.

• We build the solution vectors
$$\vec{x}_1 = \begin{bmatrix}x_{11}\\x_{21}\\\vdots\\x_{n1}\end{bmatrix},\ \vec{x}_2 = \begin{bmatrix}x_{12}\\x_{22}\\\vdots\\x_{n2}\end{bmatrix},\ \ldots,\ \vec{x}_n = \begin{bmatrix}x_{1n}\\x_{2n}\\\vdots\\x_{nn}\end{bmatrix}$$

to $A\vec{x}_1 = \vec{e}_1$, $A\vec{x}_2 = \vec{e}_2$, ..., $A\vec{x}_n = \vec{e}_n$, resp.

Then, $X := \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}$ satisfies

$$AX = A\begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} = \begin{bmatrix} A\vec{x}_1 & \cdots & A\vec{x}_n \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{bmatrix} = I_n$$

NOTE: We still have to check $XA = I_n$ as well, but we will see next time that this is automatic.

Q: What is $RE(A)$ for A of size $n \times n$ that has unique solutions to any system $A.\vec{x} = \vec{b}$?

Proposition 3: $RE(A) = I_n$, in particular $A \sim_{row} I_n$.

Proof: $r = \text{rank}(A) = \#$ non-zero rows of A. $\leq n = \#\text{rows}(A)$

- If $r < n$, then we have at least 1 free parameter for $A\vec{x} = \vec{0}$, so we cannot have unique solutions.
- Conclusion: $r = n = \#$ columns of A. & so every step of the staircase for $[A|\vec{0}]$ has length 1 . .

So $RE(A) = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & 0 & \cdots \\ & 0 & 1 & \\ & & \ddots & \ddots \\ 0 & & 0 & 1 \end{bmatrix} = I_n$

This leads to an algorithm for computing A^{-1}!

We need to solve all n systems $A\vec{x}_1 = \vec{e}_1$, $A\vec{x}_2 = \vec{e}_2$, ... $A\vec{x}_n = \vec{e}_n$

We can do this simultaneously!

§4. Algorithm for computing A^{-1}:

① Form the $n \times 2n$ matrix $[A \mid \vec{e}_1 \mid \vec{e}_2 \mid \cdots \mid \vec{e}_n] = [A \mid I_n]$

② Use Gauss-Jordan elimination to get A into reduced form

$$[A \mid I_n] \xrightarrow[\text{row operations}]{} [RE(A) \mid B] \underset{\text{Prop 3}}{=} [I_n \mid B]$$

Conclude: Columns of B solve the n original systems

$$A \, Col_1(B) = \vec{e}_1 \quad , \ldots , \quad A \, Col_n(B) = \vec{e}_n$$

In particular $\quad AB = I_n$.

Next Time: We'll see that $B = A^{-1}$, ie B also satisfies $BA = I_n$.

Idea: Since $[A \mid I_n] \underset{\text{row}}{\sim} [I_n \mid B]$ then $I_n \underset{\text{row}}{\sim} B$, and this will say B is invertible.

The same row operations (in reverse order) will give $[B \mid I_n] \longrightarrow [I_n \mid A]$

forcing $BA = I_n$.

Example: $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ Let's compute A^{-1}.

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_1]{} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to -R_2]{} \left[\begin{array}{ccc|ccc} 1 & 3 & \boxed{1} & 1 & 0 & 0 \\ 0 & 1 & \boxed{1} & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - R_3 \\ R_1 \to R_1 - R_3}]{} \left[\begin{array}{ccc|ccc} 1 & \boxed{3} & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

← fix this

← fix this!

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$$\xrightarrow{R_1 \to R_1 - 3R_2} \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{= I_3} & \underbrace{\begin{matrix} -5 & 3 & 2 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{matrix}}_{= A^{-1}} \end{array}\right]$$

Check:

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & 3 & 2 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5+6 & 3-3 & 2-3+1 \\ -10+10 & 6-5 & 4-5+1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Also

$$\begin{bmatrix} -5 & 3 & 2 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5+6+0 & -15+15 & -5+3+2 \\ 2-2 & 6-5 & 2-1-1 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Q: How did we know A was invertible?

A We didn't! So far, our only test for invertibility is:

$$A \rightsquigarrow RE(A) = I_n \, ? \quad \begin{cases} \xrightarrow{\text{YES}} & A \text{ invertible} \\ \xrightarrow{\text{NO}} & A \text{ is NOT invertible} \end{cases}$$

Later in the course we will have a different test, via determinants

$\det(A)$ is a number: • $\det(A) \neq 0$ means A is invertible

(A $n \times n$ matrix) • $\det(A) = 0$ ______ is NOT ______.