Lecture IX §1.7 Singular matrices & linear independence

Recall: Last time we defined invertible matrices & showed how to compute them.

- An $n\times n$ matrix A is invertible if there exists another $n\times n$ matrix B such that $\boxed{AB = I_n = BA}$ (*), where $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$ of size $n\times n$.

- **Properties:** The following statements are equivalent

① A is invertible

② $A \sim_{row} I_n$ ($RE(A) = I_n$)

③ All linear systems $A\vec{x} = \vec{b}$ (as we vary $\vec{b}$) have unique solns

- **Algorithm:** $[A \mid I_n] \xrightarrow[\text{(Gauss - Jordan)}]{\text{row operations}} [RE(A) \mid B]$

  - If $RE(A) \neq I_n$, then A is not invertible
  - If $RE(A) = I_n$, then B satisfies $AB = I_n$.
  - **Claim:** $BA = I_n$ as well, so $B = A^{-1}$.

Idea: Since $[A \mid I_n] \sim_{row} [I_n \mid B]$ then $I_n \sim_{row} B$, and this will say B is invertible.

The same row operations (in reverse order) will give $[B \mid I_n] \longrightarrow [I_n \mid A]$ forcing $BA = I_n$.

§1. Algebraic properties of matrix inverses:

Theorem: Fix A, C two $n\times n$ invertible matrices. Then

① $I_n^{-1} = I_n$ (since $I_n I_n = I_n$)

② $(A^{-1})^{-1} = A$ (since $A^{-1}A = AA^{-1} = I_n$, so A is the inverse of A^{-1})

③ $(AC)^{-1} = C^{-1}A^{-1}$ (since $(AC)(C^{-1}A^{-1}) = A(\underbrace{CC^{-1}}_{I_n})A^{-1} = AA^{-1} = I_n$

$(C^{-1}A^{-1})(AC) = C^{-1}(\underbrace{A^{-1}A}_{=I_n})C = C^{-1}C = I_n$)

④ $(A^T)^{-1} = (A^{-1})^T$ (since $(A^T)(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$

$(A^{-1})^T A^T = (AA^{-1})^T = I_n^T = I_n$)

⑤ a scalar $\alpha \neq 0$, then $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ (since $\alpha A(\frac{1}{\alpha}A^{-1}) = \alpha \frac{1}{\alpha} AA^{-1} = I_n$)

TODAY: Alternative characterization of invertible matrices.

§2. 2×2 invertible matrices:

Theorem: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if, and only if, $\overset{\det(A)}{\Delta} = ad - bc \neq 0$

Furthermore, $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Why? Assume $\Delta \neq 0$ & check the formula for A^{-1} works

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \Delta & -ab+ab \\ -cd+cd & \Delta \end{bmatrix} = I_2$
- $\frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \Delta & db-bd \\ -ca+ac & \Delta \end{bmatrix} = I_2$

Conversely, assume $\Delta = ad - bc = 0$. We show that $A\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ doesn't have a unique solution. If $A = \mathbb{O}$, this is clear, so we can assume $A \neq \mathbb{O}$

(1) CASE 1: $[c, d] \neq [0, 0]$

$$\vec{A} \begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} \Delta \\ cd-dc \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{so } \begin{bmatrix} d \\ -c \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

↪ not the zero vector

also solves the system.

(2) CASE 2: $[c, d] = [0, 0]$ (so both $c = 0$ & $d = 0$)

Since $A \neq \mathbb{O}$, we know $[a, b] \neq [0, 0]$. Then $A \begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} -ab+ab \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

& we conclude $\begin{bmatrix} -b \\ a \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ also solves the system.

§3. Linear dependence / independence: Fix m, n positive integers

Assume we are given a set of n vectors in $\mathbb{R}^m$, say $\{\vec{v}_1, \ldots, \vec{v}_n\}$

Definition: We say this set of vectors is <u>linearly dependent</u> if we can find n numbers $x_1, x_2, \ldots, x_n$ in $\mathbb{R}$ NOT all zero, such that

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n = \vec{0} \qquad \left(\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^m\right)$$

Otherwise, we say $\vec{v}_1, \ldots, \vec{v}_n$ are <u>linearly independent</u>. This means that the <u>only</u> solution to $x_1 \vec{v}_1 + \cdots + x_n \vec{v}_n = \vec{0}$ is $x_1 = x_2 = \cdots = x_n = 0$.

<u>Abbreviation</u>: l.i.

Examples: (1) $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is linearly dependent

$$1 \vec{v}_1 + 0 \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (x_1 = 1, x_2 = 0 \text{ is a solution})$$

(2) $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is linearly independent

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 = x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_1 + x_2 \\ -x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gives the system $\begin{cases} x_1 + x_2 = 0 \\ 2x_1 + x_2 = 0 \\ -x_2 = 0 \end{cases}$ ⇝ augmented matrix $\begin{array}{c} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right] \\ \begin{array}{cc} \overset{\shortparallel}{\vec{v}_1} & \overset{\shortparallel}{\vec{v}_2} \end{array} \end{array}$

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & -1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{} \begin{bmatrix} 1 & 1 \\ 0 & -3 \\ 0 & -1 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & -3 \end{bmatrix} \xrightarrow[R_2 \to -R_2]{} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -3 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3R_2]{} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

<u>Conclude</u> $x_1 = 0$, $x_2 = 0$ so $\{\vec{v}_1, \vec{v}_2\}$ is l.i.

Note: Write $A = [\vec{v}_1, \dots, \vec{v}_n]$ for the $m \times n$ matrix with cols $\vec{v}_1, \dots, \vec{v}_n$

- $\{\vec{v}_1, \dots, \vec{v}_n\}$ l.i $\iff$ The system $A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}$ has unique solution

- $\{\vec{v}_1, \dots, \vec{v}_n\}$ l.d $\iff$ The system $A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}$ has infinitely many solutions.
(trivial soln is NOT the only one)

Exercise: Determine whether the following 3 vectors in $\mathbb{R}^3$ are linearly independent or dependent.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

Solution: Write the system $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ as

$$\underbrace{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ -1 & 3 & 7 \end{bmatrix}}_{=A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ 3 eqns & 3 variables

- Put A in reduced echelon form:

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ -1 & 3 & 7 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \\ 0 & 6 & 12 \end{bmatrix} \xrightarrow{R_2 \to \frac{R_2}{-2}} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & 6 & 12 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 6R_2}$$

$$\underset{\text{echelon}}{\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}} \xrightarrow{R_1 \to R_1 - 3R_2} \underset{\text{red. ech.}}{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$$

x_1, x_2 dependent
x_3 independent

Solutions: ?

$$x_1 - x_3 = 0$$
$$x_2 + 2x_3 = 0$$

gives $\begin{cases} x_1 = t \\ x_2 = -2t \\ x_3 = t \end{cases}$ t free

$t=1$ gives a nontrivial solution so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dep.

A Dependency relation $\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0} \rightsquigarrow \vec{v}_1 = 2\vec{v}_2 - \vec{v}_3$ dependent linearly on $\vec{v}_2$ & $\vec{v}_3$.

Theorem: If we are given n vectors in $\mathbb{R}^m$ & $n > m$, then the vectors are linearly dependent.

Why? We get a homogeneous system $A\vec{x} = \vec{0}$ with A has size $m \times n$, so we have # eqns < # variables, so the solution $\vec{x}$ cannot be unique. This means the vectors are l.d.

§4. Singular matrices:

Definition: An $n \times n$ matrix $A = (a_{ij})$ is called nonsingular if $A\vec{x} = \vec{0}$ has only one solution (namely the trivial one: $\vec{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$)

On the other hand, if the solution is NOT unique, we say A is singular

Observation 1: A is non-singular $\longleftrightarrow$ Columns of A are l.i. vectors in $\mathbb{R}^n$

A is singular $\longleftrightarrow$ Columns of A are l. dep.

Observation 2: If A invertible, then A is nonsingular because the system $A\vec{x} = \vec{0}$ (choose $\vec{b} = \vec{0}$) has a unique solution.

The converse of this statement is also true because $A\vec{x} = \vec{0}$ has a unique solution if and only if all n variables are dependent ($\text{rank}(A) = n$)

The size of A forces $RE(A) = I_n$, so A is invertible.

Theorem (Summary) The following statements mean one and the same thing for any given matrix $A = (a_{ij})$ of size $n \times n$.

(1) A is invertible

(2) Rank of $A = n$

(3) Reduced echelon form of A is I_n

(4) A is nonsingular

(5) Columns of A are n linearly independent vectors in $\mathbb{R}^n$.