Lecture X: §2.1 Vectors in the plane
§2.2 Vectors in space

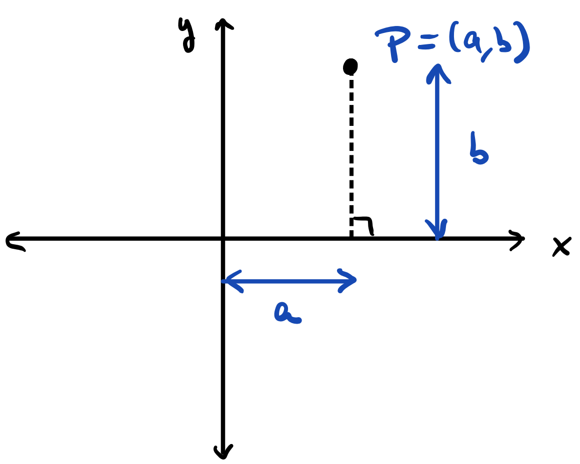
§1. Introduction

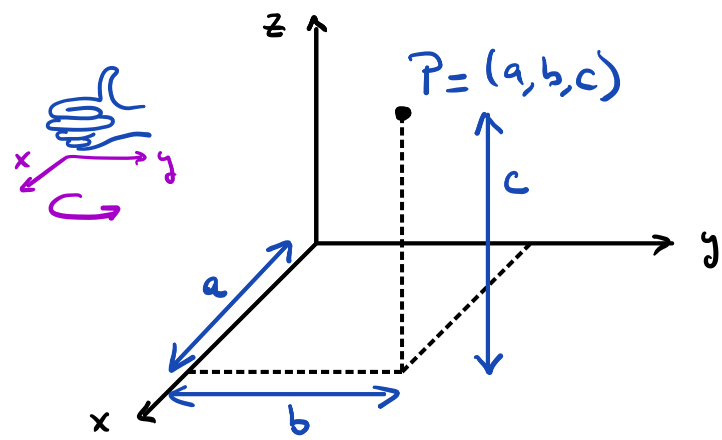
Up to now, we've describe Euclidean space $\mathbb{R}^n$ as the space of $n\times 1$ matrices, with addition, scalar multiplication and dot product operation. All these operations are defined algebraically & they are well behaved.

- TODAY'S FOCUS: For $n=2$ & 3, vectors can be drawn & these operations have clear geometric meaning.
- Historically speaking, the idea of vectors comes from Physics (~1690's) after Newton's laws of motion were written down. The first "vector" was force acting on a particle, where in order the accurately apply Newton's Principles we need to know not only the "magnitude" but also the "direction" in which the force is acting.

§2. Coordinate systems in 2 & 3 dimensions: (cartesian coordinates)

Definition: A point in $\mathbb{R}^2$ is specified by 2 real numbers.
— $\mathbb{R}^3$ — 3 —

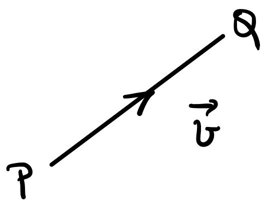




- Right-hand rule for (x,y,z)-coordinates in $\mathbb{R}^3$: If you place your right hand along the x-axis in such a way that your fingers can move towards the y-axis, then your thumb will put in the z-direction

§3. Vectors in $\mathbb{R}^2$ & $\mathbb{R}^3$:

Geometrically, a vector is represented as a directed line segment joining two points P & Q.

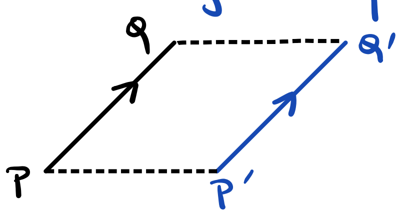


- starting point (or tail of $\vec{v}$) = P
- terminal point (or head of $\vec{v}$) = Q

Sometimes we write $\vec{v} = \overrightarrow{PQ}$

- Length of $\vec{v}$ = magnitude

Observations: (1) We don't distinguish between parallel line segments of the same length & pointing in the same direction

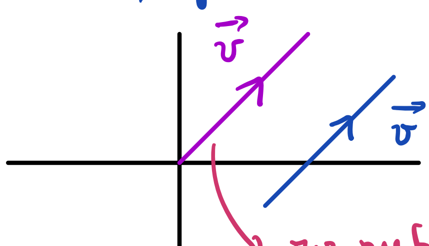


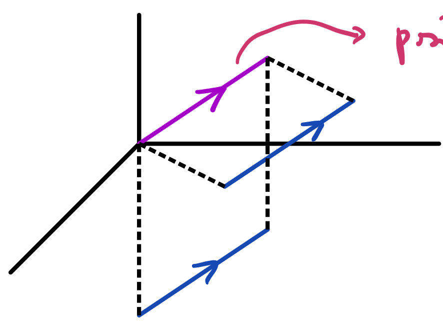
$$\overrightarrow{PQ} = \overrightarrow{P'Q'}$$

(2) $\vec{0}$ = the only vector of magnitude 0 and no direction
$= \overrightarrow{PP}$ for any point P.

Q: Can we have a consistent choice to draw vectors?

A: Yes, place the tail at the origin ((0,0) or (0,0,0))





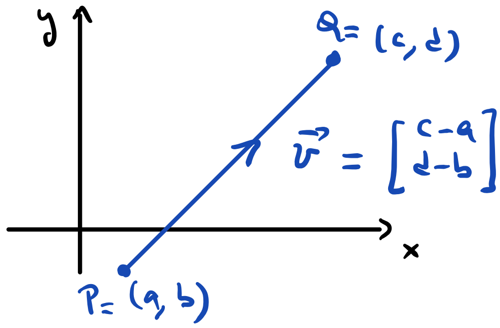
Name: position vector

After this choice is made, we identify the position vector with the location of its head P, i.e. $\vec{v} = \overrightarrow{OP}$. The coordinates of P are the components of $\vec{v}$

- As a 2×1 matrix, a vector in $\mathbb{R}^2$ is represented geometrically by joining $P = (a,b)$ to $Q = (c,d)$ in the (x,y)-plane is given by

$$\overrightarrow{PQ} = \begin{bmatrix} c-a \\ d-b \end{bmatrix}$$

x-component $= c-a$
y-component $= d-b$



- Similarly in 3-dimensions: if $P = (a_1, b_1, c_1)$ & $Q = (a_2, b_2, c_2)$, then

$$\overrightarrow{PQ} = \begin{bmatrix} a_2 - a_1 \\ b_2 - b_1 \\ c_2 - c_1 \end{bmatrix}$$

("difference of coordinates")

x-component $= a_2 - a_1$; y-component $= b_2 - b_1$; z-component $= c_2 - c_1$

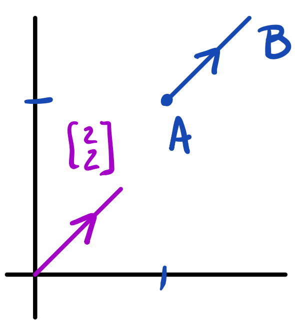
Example: If $P = (1,1,1)$ and $\overrightarrow{PQ} = \vec{v} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, then

$Q = (a,b,c)$ satisfies $\begin{matrix} a-1 = 2 \\ b-1 = -1 \\ c-1 = 4 \end{matrix}$ $\rightsquigarrow$ $Q = (3, 0, 5)$

Q Why do we like position vectors?

A We can decide when two vectors agree: $\vec{v} = \vec{w}$ if their position vectors agree

Example: Given $\overrightarrow{AB}$ with position vector $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, find the coordinates of B given $A = (3,4)$ & draw $\overrightarrow{AB}$.



$B = (b_1, b_2)$ $\qquad$ $A = (3,4)$

$$\vec{v} = \overrightarrow{AB} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} b_1 - 3 \\ b_2 - 4 \end{bmatrix}$$

So $\begin{matrix} 2 = b_1 - 3 \\ 2 = b_2 - 4 \end{matrix}$ gives $B = (2+3, 2+4) = (5,6)$

• Magnitude of $\vec{v}$ = ? If $\vec{v} = \overrightarrow{PQ}$, then $\|\vec{v}\|$ is the distance from P to Q. Algebraically:

. $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$ $\rightsquigarrow$ $\|\vec{v}\| = \sqrt{a^2+b^2}$

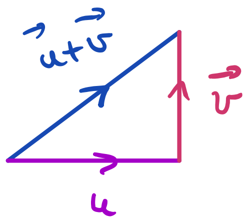
. $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ $\rightsquigarrow$ $\|\vec{v}\| = \sqrt{a^2+b^2+c^2}$

§4. Addition & Scalar Multiplication

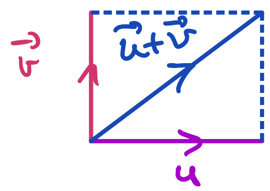
• Algebraically: Vectors are viewed as 2x1 or 3x1 matrices and scaled & added componentwise.

Example: $5\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + 3\begin{bmatrix} -2 \\ -4 \\ -7 \end{bmatrix} = \begin{bmatrix} 5\cdot 2 + 3(-2) \\ 5\cdot 1 + 3(-4) \\ 5\cdot(-1) + 3\cdot 7 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 16 \end{bmatrix}$

• Geometrically: We have "Triangle & parallelogram laws" for vector addition:



Triangle Law

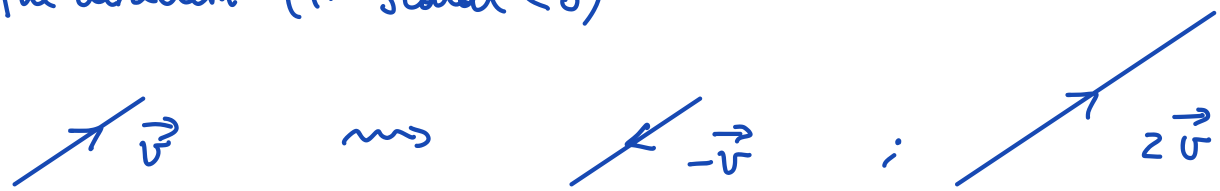


Parallelogram Law

① Triangle Law: Move $\vec{v}$ parallel to itself until Tail of $\vec{v}$ = Head of $\vec{u}$. Then $\vec{u}+\vec{v}$ starts at the tail of $\vec{u}$ and ends at the head of the translated $\vec{v}$.

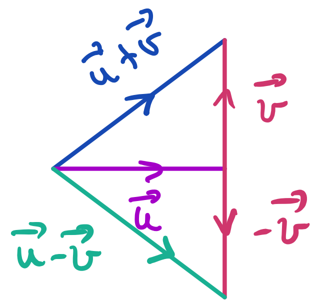
② Parallelogram Law: If $\vec{u}$ & $\vec{v}$ start at the same point, complete the parallelogram. Then $\vec{u}+\vec{v}$ starts at the common tail and ends at the opposite corner in the parallelogram.

- For scalar multiplication, we have to change the length of the vector while either retaining the same direction (if scalar<0) or flipping the direction (if scalar<0)

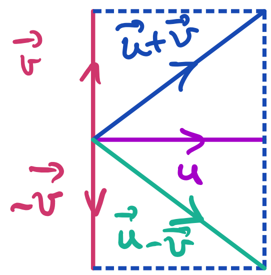


- For substraction or difference: combine addition & scalar mult

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}).$$



Triangle Law

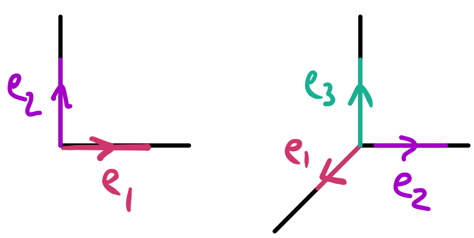


Parallelogram Law

§5. Coordinate Vectors (or basic unit vectors)

$\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ in $\mathbb{R}^2$; $\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ in $\mathbb{R}^3$

In Calculus III, we use $\vec{i}, \vec{j}, \vec{k}$ for these, and write vectors as pairs or triples of numbers surrounded by angular brackets



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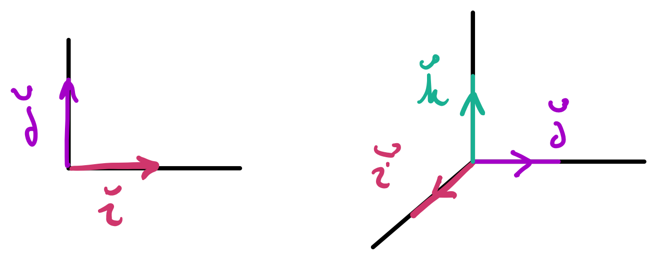

$\begin{bmatrix}1\\-1\\4\end{bmatrix}$ in this course vs $\langle 1,-1,4\rangle$ in Calculus III.

Proposition: Any vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a linear combination of basic unit vectors

Why? $\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Definition: A <u>unit vector</u> is any vector of length 1.

Example: Find <u>the</u> unit vector in the same direction as $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \rightsquigarrow \|\vec{v}\| = \sqrt{1^2+4^2+7^2} = \sqrt{1+16+49} = \sqrt{66}$

Answer: $\vec{u} = \text{unit vector} = \dfrac{1}{\|\vec{v}\|}\vec{v} = \begin{bmatrix} 1/\sqrt{66} \\ 4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$

Example: Same question, but now require length 8 & opposite direction to $\vec{v}$

Soln: $\vec{w} = -8 \cdot \vec{u} = \dfrac{-8}{\|\vec{v}\|}\vec{v} = \begin{bmatrix} -8/\sqrt{66} \\ -32/\sqrt{66} \\ -56/\sqrt{66} \end{bmatrix}$