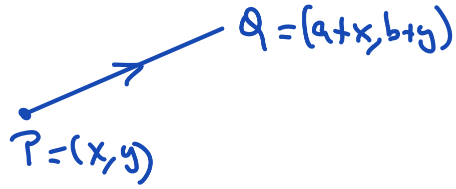
Lecture XI: § 2.3 The dot product

Recall Last Time we discussed vectors in 2 & 3 dimensions geometrically

Algebraically: $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$; $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$

Geometrically



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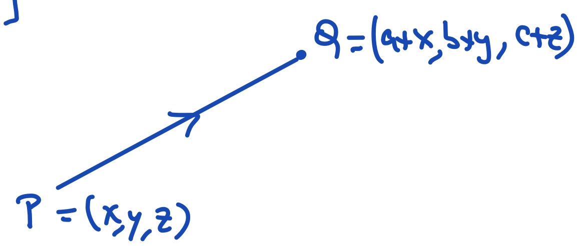
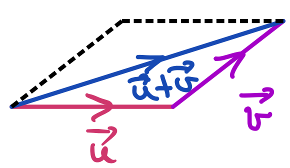


Magnitude of $v = \|\vec{v}\| = \text{distance}(P, Q)$

$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \rightsquigarrow \|\vec{v}\| = \sqrt{a^2 + b^2}$ in $\mathbb{R}^2$

$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightsquigarrow \|\vec{v}\| = \sqrt{a^2 + b^2 + c^2}$ in $\mathbb{R}^3$

Geometric interpretation of addition & scalar multiplication.



Triangle / Parallelogram Law

$\|a\vec{v}\| = |a| \, \|\vec{v}\|$ for a in $\mathbb{R}$

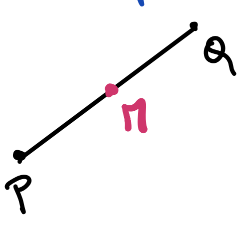
Direction of $a\vec{v}$ = $\begin{cases} \text{same as } \vec{v} & \text{if } a > 0 \\ \text{opposite to } \vec{v} & \text{if } a < 0 \\ \text{no direction} & \text{if } a = 0 \\ (0 \cdot \vec{v} = \vec{0}) & \end{cases}$

- Basic unit vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$

  $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$

- Given $\vec{v} \neq \vec{0}$, the is a unique unit vector in the direction of $\vec{v}$, namely $\frac{1}{\|\vec{v}\|}\vec{v}$

Q Midpoint between 2 points P & Q in $\mathbb{R}^3$?



$P = (p_1, p_2, p_3)$

$Q = (q_1, q_2, q_3)$

$M = (m_1, m_2, m_3)$

M has 2 properties

- M in the segment joining P & Q
- $\|\vec{PM}\| = \|\vec{MQ}\| = \frac{1}{2}\|\vec{PQ}\|$

$\vec{PM} = a\,\vec{PQ}$ for some $a>0$ (same direction as $\vec{PQ}$) & $\|\vec{PM}\| = \|a\vec{PQ}\| = |a|\,\|\vec{PQ}\|$

forces $a = \frac{1}{2}$ $\qquad = a\|\vec{PQ}\| = \frac{1}{2}\|\vec{PQ}\|$

$\rightsquigarrow (m_1, m_2, m_3) = (p_1, p_2, p_3) + \frac{1}{2}(q_1-p_1, q_2-p_2, q_3-p_3) = \left(\frac{p_1+q_1}{2}, \frac{p_2+q_2}{2}, \frac{p_3+q_3}{2}\right)$

§1. Dot Product:

Recall Given $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$ in $\mathbb{R}^m$, we define the

dot product $\vec{u}\cdot\vec{v} = \vec{u}^T\,\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_mv_m$ (a number!)

Example $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$, $\vec{u}\cdot\vec{v}$ $1\cdot 0 + (-1)\cdot 3 + 1\cdot 5 = -3+5 = 2$

By construction, dot products inherit several properties from matrix addition, scalar multiplication & product. More precisely:

Algebraic properties: Fix $\vec{u}, \vec{v}, \vec{w}$ vectors in $\mathbb{R}^m$, a = scalar

1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ (not true for general matrices!)

2. [Distributive] $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w} = \vec{w}\cdot(\vec{u}+\vec{v})$

3. [Associative] $(a\vec{u})\cdot\vec{v} = a(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(a\vec{v})$

4. $\vec{u}\cdot\vec{u} = \|\vec{u}\|^2$

Application: $\|\vec{u}+\vec{v}\|^2 = ?$ in terms of $\vec{u}$ & $\vec{v}$

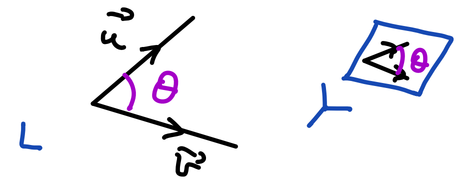
$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \underset{\text{Distrib } (2)}{=} \vec{u}\cdot\vec{u} + \underbrace{\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u}}_{= (1)} + \vec{v}\cdot\vec{v}$$

$$\underset{(4)}{=} \|\vec{u}\|^2 + 2\,\vec{u}\cdot\vec{v} + \|\vec{v}\|^2.$$

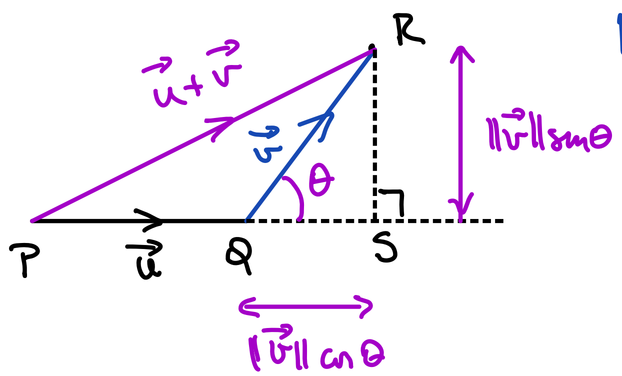
Q: Can we compute $\vec{u}\cdot\vec{v}$ in $\mathbb{R}^2$ & $\mathbb{R}^3$ using Geometry? A: YES

- Geometric formula for $\vec{u}\cdot\vec{v}$: If θ = angle between $\vec{u}$ & $\vec{v}$ $(0 \le \theta \le \pi)$

then $\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta}$



Why? We compute $\|\vec{u}+\vec{v}\|^2$ geometrically and compare it with the formula written in the application above.



$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= \|\vec{PR}\|^2 \\
&= \|\vec{PS}\|^2 + \|\vec{SR}\|^2 \\
&= \left(\|\vec{u}\| + \|\vec{v}\|\cos\theta\right)^2 + \left(\|\vec{v}\|\sin\theta\right)^2 \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} \\
&\quad + 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta
\end{aligned}$$

So $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta$

But $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\,\vec{u}\cdot\vec{v}$

Conclude: $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$

Examples (1) Pick $\vec{u}, \vec{v}$ of length 3 & 6 respectively with angle $\theta = 60°$ between them. Find $\vec{u}\cdot\vec{v}$.

Soln $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos 60° = 3\cdot 6\cdot \frac{1}{2} = \boxed{9}$

(2) Find the angle between $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$

Soln: $\|\vec{u}\| = \sqrt{2^2+1^2} = \sqrt{5}$

$\|\vec{v}\| = \sqrt{1^2+1^2+5^2} = \sqrt{27}$

$\vec{u}\cdot\vec{v} = 2\cdot(-1) + 1\cdot 1 + 0\cdot 5 = -1$

$\rightsquigarrow \cos\theta = \frac{-1}{\sqrt{5}\sqrt{27}} = \frac{-1}{3\sqrt{15}} \rightsquigarrow \theta \approx 95°$

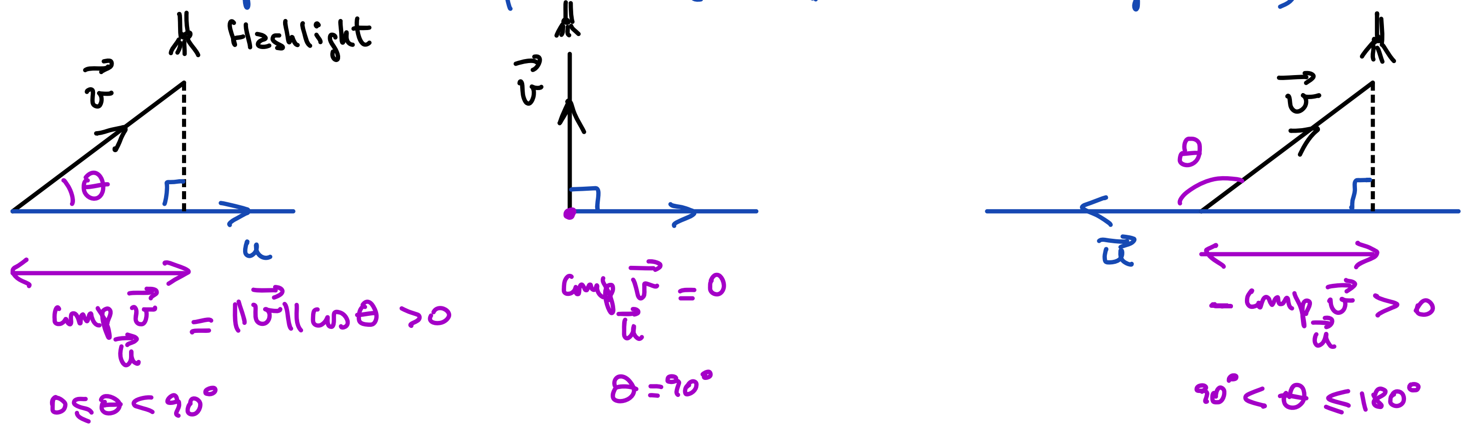
§2. Orthogonal vectors & projection:

Definition: Two vectors $\vec{u}$ & $\vec{v}$ are orthogonal (or perpendicular) if

$$\vec{u}\cdot\vec{v} = 0 \qquad (\theta \text{ between them is } 0 \text{ since } \cos\theta = 0)$$

We denote this by $\vec{u} \perp \vec{v}$.

- In Physical problems, we sometimes have to compute the "component of $\vec{v}$ along $\vec{u}$" or the "projection of $\vec{v}$ onto $\vec{u}$". This construction will be later use to build orthogonal bases for $\mathbb{R}^n$ (Gramm-Schmidt Algorithm)



- Component of $\vec{v}$ along $\vec{u}$ $= \text{comp}_{\vec{u}}\vec{v} = \|\vec{v}\|\cos\theta$

$$= \cancel{\|\vec{v}\|}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\cancel{\|\vec{v}\|}}\right)$$

Conclude: $\boxed{\text{comp}_{\vec{u}}\vec{v} = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}}$ (signed length)

- Projection of $\vec{v}$ onto $\vec{u}$ $= \begin{pmatrix}\text{component of } \vec{v} \\ \text{along } \vec{u}\end{pmatrix} \cdot \begin{pmatrix}\text{unit vector} \\ \text{along } \vec{u}\end{pmatrix}$

$$= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|} \cdot \frac{\vec{u}}{\|\vec{u}\|}$$

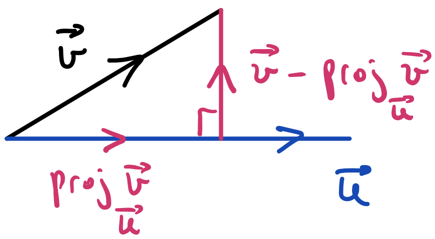
Conclude: $\boxed{\text{proj}_{\vec{u}}\vec{v} = \dfrac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u}}$

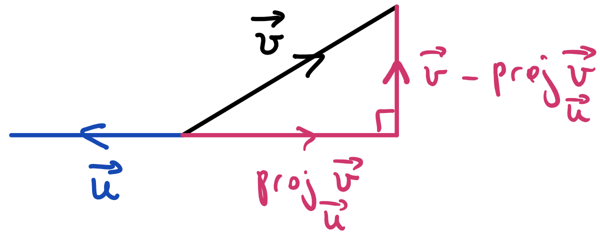
Q: Why is this important? **A** We can decompose $\vec{v}$ as a sum $\vec{w}+\vec{s}$

$$\vec{v} = \vec{w} + \vec{s}$$ where $\vec{w} \parallel \vec{u}$ & $s \perp \vec{u}$

$$\boxed{\vec{v} = \text{proj}_{\vec{u}}\vec{v} + (\vec{v} - \text{proj}_{\vec{u}}\vec{v})}$$

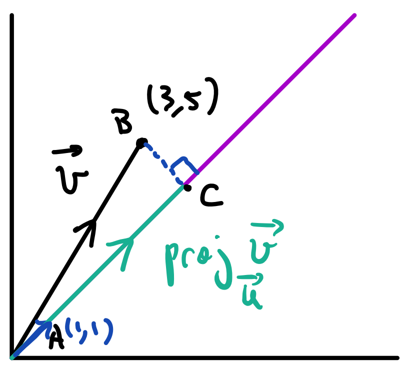
(these is the only option for $\vec{w}$ & $\vec{s}$)





Example 1: Fix $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ Compute the projection of $\vec{v}$ onto $\vec{u}$

$$\underset{=\vec{OC}}{\text{proj}_{\vec{u}}\vec{v}} = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u} = \frac{1\cdot 3 + 1\cdot 5}{1^2+1^2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{8}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$



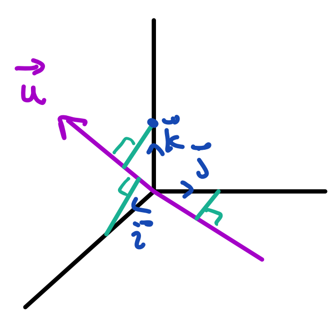
Example 2: Compute the projection of $\vec{e}_1$, $\vec{e}_2$ & $\vec{e}_3$ along $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

- $\vec{u}\cdot\vec{u} = \left\| \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\|^2 = 1^2 + (-1)^2 + 2^2 = 6$, $\vec{u}\cdot\vec{e}_1 = 1$, $\vec{u}\cdot\vec{e}_2 = -1$, $\vec{u}\cdot\vec{e}_3 = 2$

$$\text{proj}_{\vec{u}} e_1 = \frac{\vec{u}\cdot\vec{e}_1}{\vec{u}\cdot\vec{u}}\,\vec{u} = \frac{1}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ -1/6 \\ 1/3 \end{bmatrix}$$

$$\text{proj}_{\vec{u}} e_2 = \frac{\vec{u}\cdot\vec{e}_2}{\vec{u}\cdot\vec{u}}\,\vec{u} = \frac{-1}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/6 \\ 1/6 \\ -1/3 \end{bmatrix}$$

$$\text{proj}_{\vec{u}} e_3 = \frac{\vec{u}\cdot\vec{e}_3}{\vec{u}\cdot\vec{u}}\,\vec{u} = \frac{2}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$



§3. Cross product in $\mathbb{R}^3$:

Fix $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in $\mathbb{R}^3$

Definition: The cross product $\vec{u} \times \vec{v}$ is a vector in $\mathbb{R}^3$ with coordinates

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -(u_1 v_3 - u_3 v_1) \\ u_1 v_2 - u_2 v_2 \end{bmatrix}$$

Note: Each component is a 2x2 determinant $\left[\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \doteq \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \right]$

$$\det\begin{pmatrix} x & y & z \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = x \underbrace{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}}_{\text{remove row 1, col 1}} - y \underbrace{\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}}_{\text{remove row 1, col 2}} + z \underbrace{\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}_{\text{remove row 1, col 3}}$$

So $$\vec{u} \times \vec{v} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = \underbrace{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}}_{\text{scalar}} \vec{i} - \underbrace{\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}}_{\text{scalar}} \vec{j} + \underbrace{\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}_{\text{scalar}} \vec{k}$$

Example $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$$\vec{u} \times \vec{v} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{pmatrix} = \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \vec{k}$$

$$= (4+3)\vec{i} - (2-6)\vec{j} + (-1-4)\vec{k} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$$

Properties $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$, a, b scalars

(1) [Anticommutative] $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

In particular, $\vec{u} \times \vec{u} = -\vec{u} \times \vec{u} = \vec{0}$.

② [Associative] $(a\vec{u}) \times (b\vec{v}) = (ab)\ \vec{u} \times \vec{v}$, so $\vec{0} \times \vec{u} = \vec{0}$

③ [Distributive]

$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

$$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$$

④ $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

Why ① ? $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = -\det \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

Consequence of ④: $\vec{u} \perp \vec{u} \times \vec{v}$ & $\vec{v} \perp \vec{u} \times \vec{v}$

Why? $\vec{u} \cdot (\vec{u} \times \vec{v}) = (\vec{u} \times \vec{u}) \cdot \vec{v} = \vec{0} \cdot \vec{v} = 0$

& $\vec{v} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (-\vec{v} \times \vec{u}) = -\vec{v} \cdot (\vec{v} \times \vec{u}) = 0$

same idea