Lecture XII: §2.3 The cross product

§3. Cross product in $\mathbb{R}^3$:

Fix $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in $\mathbb{R}^3$

Definition: The cross product $\vec{u} \times \vec{v}$ is a vector in $\mathbb{R}^3$ with coordinates

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -(u_1 v_3 - u_3 v_1) \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Note: Each component is a 2×2 determinant $\left[\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \right]$

$$\det\begin{pmatrix} x & y & z \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = x \underbrace{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}}_{\text{remove row 1, col 1}} - y \underbrace{\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}}_{\text{remove row 1, col 2}} + z \underbrace{\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}_{\text{remove row 1, col 3}}$$

So
$$\vec{u} \times \vec{v} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = \underbrace{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}}_{\text{scalar}} \vec{i} - \underbrace{\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}}_{\text{scalar}} \vec{j} + \underbrace{\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}_{\text{scalar}} \vec{k}$$

Example $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$$\vec{u} \times \vec{v} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{pmatrix} = \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \vec{k}$$
$$= (4+3)\vec{i} - (2-6)\vec{j} + (-1-4)\vec{k} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$$

Properties $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$, a, b scalars

1. [Anticommutative] $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ $\left(\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\det\begin{pmatrix} c & d \\ a & b \end{pmatrix} \right)$

   In particular, $\vec{u} \times \vec{u} = -\vec{u} \times \vec{u} = \vec{0}$.

② [Associative] $(a\vec{u}) \times (b\vec{v}) = (ab)\, \vec{u} \times \vec{v}$, so $\vec{0} \times \vec{u} = \vec{0}$

③ [Distributive]

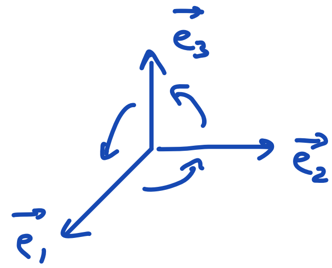
$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

$$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$$

④ $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ (direct calculation)

Why ① ? $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = -\det\begin{pmatrix} c & d \\ a & b \end{pmatrix}$

Alternative way to compute:



(1) Know $\vec{e_1} \times \vec{e_2}$, $\vec{e_1} \times \vec{e_3}$, $\vec{e_2} \times \vec{e_3}$:

$$\vec{e_1} \times \vec{e_2} = \vec{e_3}$$

$$\vec{e_2} \times \vec{e_3} = \vec{e_1}$$

$$\vec{e_3} \times \vec{e_1} = \vec{e_2}$$

(2) Use Properties ①, ② & ③ to compute

Example $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix}$ Claim $\vec{u} \times \vec{v} = -21\,\vec{e_1} + 12\vec{e_2} + 11\vec{e_3}$

$\vec{u} = \vec{e_1} - \vec{e_2} + 3\vec{e_3}$, $\vec{v} = 4\vec{e_1} - 7\vec{e_2}$

$$\vec{u} \times \vec{v} = (\vec{e_1} - \vec{e_2} + 3\vec{e_3}) \times (4\vec{e_1} + 7\vec{e_2})$$

$$\overset{③}{=} \vec{e_1} \times 4\vec{e_1} + \vec{e_1} \times 7\vec{e_2} - \vec{e_2} \times 4\vec{e_1} + \vec{e_2} \times 7\vec{e_2} + 3\vec{e_3} \times 4\vec{e_1} + 3\vec{e_3} \times 7\vec{e_2}$$

$$\overset{②}{=} 4\,\vec{e_1} \times \vec{e_1} + 7\,\vec{e_1} \times \vec{e_2} - 4\,\vec{e_2} \times \vec{e_1} + 7\vec{e_2} \times \vec{e_2} + 12\,\vec{e_3} \times \vec{e_1} + 21\,\vec{e_3} \times \vec{e_2}$$

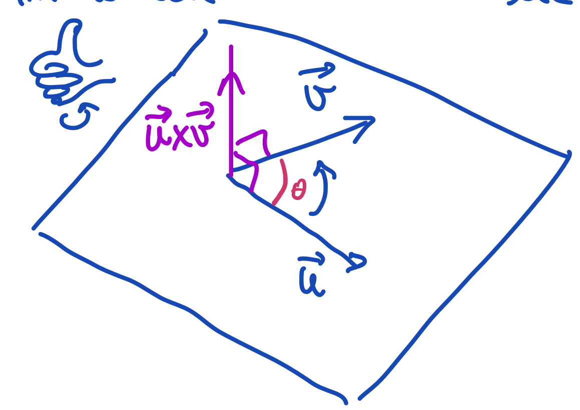
$$\overset{①}{=} 4 \cdot 0 + (7+4)\,\vec{e_1} \times \vec{e_2} + 7 \cdot 0 + 12\,\vec{e_3} \times \vec{e_1} - 21\,\vec{e_2} \times \vec{e_3} = 11\,\vec{e_3} + 12\,\vec{e_2} - 21\,\vec{e_1}$$

Consequence of ④: $\vec{u} \perp \vec{u} \times \vec{v}$ & $\vec{v} \perp \vec{u} \times \vec{v}$

Why? $\vec{u} \cdot (\vec{u} \times \vec{v}) = (\vec{u} \times \vec{u}) \cdot \vec{v} = \vec{0} \cdot \vec{v} = 0$

& $\vec{v} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (-\vec{v} \times \vec{u}) = -\vec{v} \cdot (\vec{v} \times \vec{u}) = \underset{\text{same idea}}{0}$

- Thus, given $\vec{u}$ & $\vec{v}$, the direction of $\vec{u} \times \vec{v}$ is $\perp$ to the plane in which $\vec{u}$ & $\vec{v}$ lie: it comes from the right-hand rule



- To determine $\vec{u} \times \vec{v}$ we need its magnitude. There is a geometric interpretation for this quantity (as with the dot product)

Sine Formula $\boxed{\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta}$ $0 \leq \theta \leq \pi$ angle between $\vec{u}$ & $\vec{v}$

(The angle θ can be recovered from $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$ (LAST TIME))

Why is the sine formula true? We check that

$$(*) \qquad \|\vec{u} \times \vec{v}\|^2 + (\vec{u} \cdot \vec{v})^2 = \|\vec{u}\|^2 \|\vec{v}\|^2$$

and use the cosine formula for the dot product + $\cos^2\theta = 1 - \sin^2\theta$

$$\|\vec{u} \times \vec{v}\|^2 + \|\vec{u}\|^2 \|\vec{v}\|^2 (1 - \sin^2\theta) = \|\vec{u}\| \, \|\vec{v}\|^2$$

gives $$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 \sin^2\theta$$ ($\sin\theta \geq 0$ if $0 \leq \theta \leq 180°$)

- We check $(*)$ by a direct calculation $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

(LHS) of $(*)$: $(u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2 + (u_1 v_1 + u_2 v_2 + u_3 v_3)^2$

$= u_2^2 v_3^2 + u_3^2 v_2^2 \cancel{- 2 u_2 u_3 v_2 v_3} + u_3^2 v_1^2 + u_1^2 v_3^2 \cancel{- 2 u_1 u_3 v_1 v_3} + u_1^2 v_2^2 + u_2^2 v_1^2$

$$-2u_1u_2v_1v_2 + u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 + 2u_1u_2v_1v_2 + \cancel{2u_1u_3v_1v_3} + \cancel{2u_2u_3v_2v_3}$$

$$= u_1^2(v_1^2+v_2^2+v_3^2) + u_2^2(v_1^2+v_2^2+v_3^2) + u_3^2(v_1^2+v_2^2+v_3^2) = \|\vec{u}\|^2\|\vec{v}\|^2$$

$$= (\text{RHS}) \text{ of } (*)$$

§2. Applications

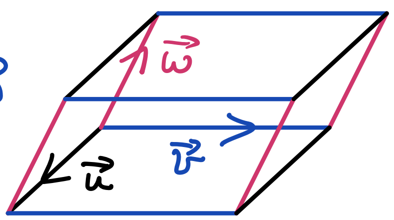
(1) Check Collinearity:

We can test whether 2 vectors lie on a line (collinear) by checking if their cross product is 0 ($\theta = 0$ or 180°)

(Easier way: check if directions are scalar multiples of each other)

(2) Volume of parallelopipe:

P



Parallelopipe with sides $\vec{u}$, $\vec{v}$ & $\vec{w}$ has volume: $\boxed{\mathrm{vol}(P) = |\vec{u} \cdot (\vec{v} \times \vec{w})|}$ (page 2)

EXAMPLE: Determine whether $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 7 \\ 3 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}$ are coplanar (ie lie on the same plane) or not.

Solution: coplanar $\equiv$ parallelopipe P is flat ($\mathrm{vol}(P) = 0$)

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 3 & 7 & 3 \\ 5 & 7 & 1 \end{vmatrix} = (7-21)i - (3-15)j + (21-35)k = \begin{bmatrix} -14 \\ 12 \\ -14 \end{bmatrix}$$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = [1 \; 0 \; -1] \begin{bmatrix} -14 \\ 12 \\ -14 \end{bmatrix} = -14 + 0 + 14 = 0,$$ so $\vec{u}, \vec{v}, \vec{w}$ are coplanar

Remarks: (1) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = (\vec{u} \times \vec{v}) \cdot \vec{w}$

(2) $(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$

Eg: $(\vec{e}_1 \times \vec{e}_2) \times \vec{e}_2 = \vec{e}_3 \times \vec{e}_2 = -\vec{e}_1$ but $\vec{e}_1 \times (\vec{e}_2 \times \vec{e}_2) = \vec{e}_1 \times \vec{0} = \vec{0}$