# Lecture XIII: §2.4 Lines and planes

**Recall:** So far, we've studied the following 3 operations for vectors in $\mathbb{R}^2$ & $\mathbb{R}^3$ from the geometric perspective

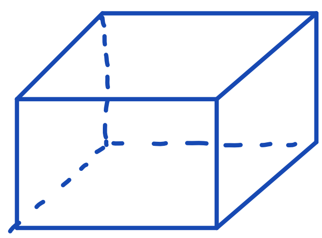
1. Addition & scalar multiplication (generalizes to $\mathbb{R}^n$)
2. Dot product (generalizes to $\mathbb{R}^n$)
3. Cross product (ONLY for $\mathbb{R}^3$)

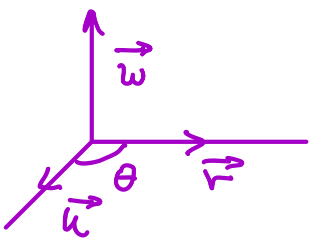
**Recall:** For $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in $\mathbb{R}^3$, the cross product $\vec{u} \times \vec{v}$ is

$$\vec{u} \times \vec{v} = \det \begin{bmatrix} \check{i} & \check{j} & \check{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = \underbrace{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}}_{\text{scalar}} \check{i} - \underbrace{\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}}_{\text{scalar}} \check{j} + \underbrace{\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}_{\text{scalar}} \check{k}$$

- $\vec{u} \times \vec{v}$ is $\perp$ to $\vec{u}, \vec{v}$ with direction given by the right-hand rule
- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$ for $\theta$ = angle between $\vec{u}$ & $\vec{v}$

**Application:** Compute volumes of parallelopipes

$\mathcal{P}$ 

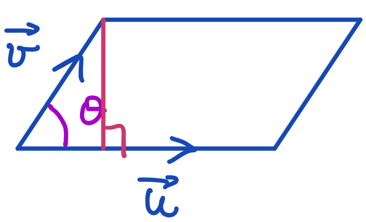



vectors forming $\mathcal{P}$: $\vec{u}, \vec{v}, \vec{w}$

$$\boxed{\mathrm{Vol}(\mathcal{P}) = |\vec{w} \cdot (\vec{u} \times \vec{v})|}$$

Why is this true? Key:
- cosine formula for dot product $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$
- sine ——— cross ———

- Area of the base of $\mathcal{P}$: Area of parallelogram formed by $\vec{u}$ & $\vec{v}$.

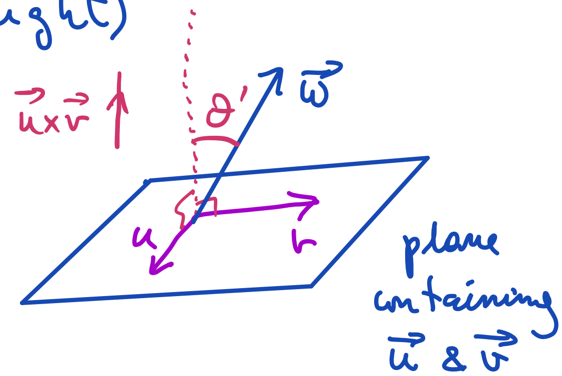



$h = \|\vec{v}\| \sin\theta$

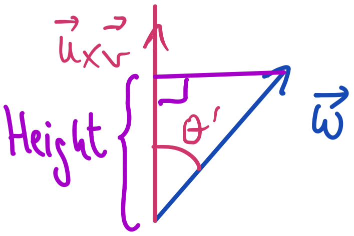
$$\text{Area}(\,\text{▱}\,) = \|\vec{u}\| \cdot h = \|\vec{u}\| \, \|\vec{v}\| \sin\theta = \|\vec{u} \times \vec{v}\|$$

• Volume $(\mathcal{P})$ = (Area of the base) × (height)

Let $\theta'$ = angle between $\vec{w}$ & $\vec{u}\times\vec{v}$




Height = $\|\vec{w}\|\,|\cos\theta'|$




$$\mathrm{Vol}(\mathcal{P}) = \underbrace{\|\vec{w}\|\,|\cos\theta'|}_{\text{Height}}\ \underbrace{\|\vec{u}\times\vec{v}\|}_{\text{area of the base}}$$

$$= |\vec{w}\cdot(\vec{u}\times\vec{v})|$$

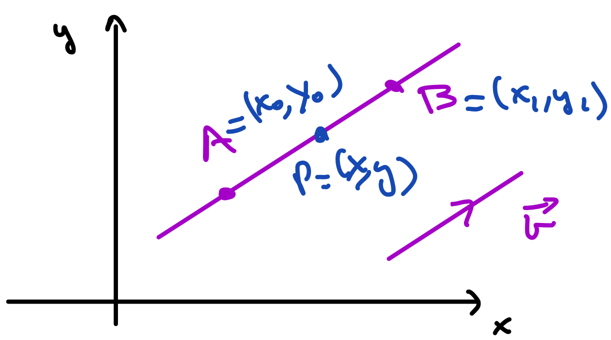
## §1. Lines in $\mathbb{R}^2$

A line in $\mathbb{R}^2$ is determined in 2 ways:

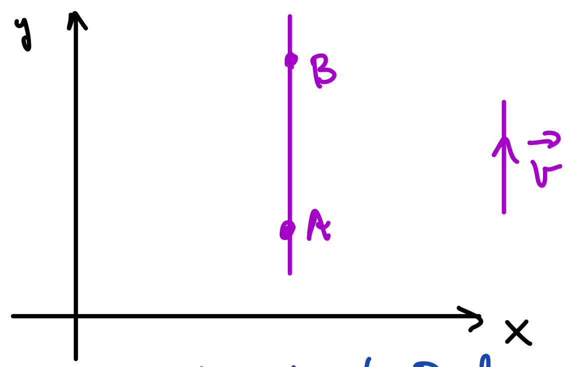
(1) data of 2 different points on it $(A, B)$

OR (2) ——— a pt on it and a direction $(A, \vec{v} = \overrightarrow{AB})$

Two different scenarios depending on $\vec{v}$:




$x_0 \neq x_1$




$x_0 = x_1$ (vertical line)

Eqn: $\boxed{y = m(x-x_0)+y_0}$

with $m = \text{slope} = \dfrac{y_1-y_0}{x_1-x_0}$

Eqn $\boxed{x = x_0}$

(slope $=\infty$)

• Both cases can be written as $\boxed{\ell : \ ax+by=c}$ for $a,b,c$ in $\mathbb{R}$ where either $a$ or $b \neq 0$.

Parametric equation: direction $= \begin{bmatrix} x_1-x_0 \\ y_1-y_0 \end{bmatrix}$

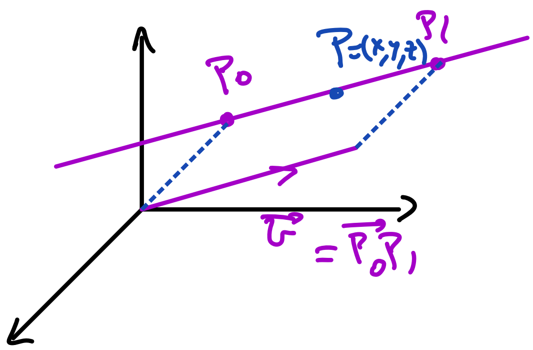
$$\overrightarrow{OP} = \overrightarrow{OA} + t\,\overrightarrow{AB} \quad \text{for some } t \text{ in } \mathbb{R}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} x_1-x_0 \\ y_1-y_0 \end{bmatrix}$$

## §2. Lines in $\mathbb{R}^3$:

A <u>line</u> $L$ in $\mathbb{R}^3$ is given by a point $P_0 = (x_0, y_0, z_0)$ in $\mathbb{R}^3$ "initial pt"
and a vector $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ ("direction vector")

$L = \{$ all points $Q$ in $\mathbb{R}^3$ where $\overrightarrow{P_0Q}$ is parallel to $\vec{v}$ $\}$
|| proportional




$P_0 = (x_0, y_0, z_0)$

$P_1 = (x_1, y_1, z_1)$

$\vec{v} = \overrightarrow{P_0P_1}$

Vector Eqn: $\overrightarrow{OP} = \overrightarrow{OP_0} + t\,\overrightarrow{P_0P_1}$ for $t$ in $\mathbb{R}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}}_{\substack{\text{initial} \\ \text{point} \\ (\text{at } t=0)}} + t \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\substack{\text{direction} \\ \text{vector}}} \qquad (*)$$

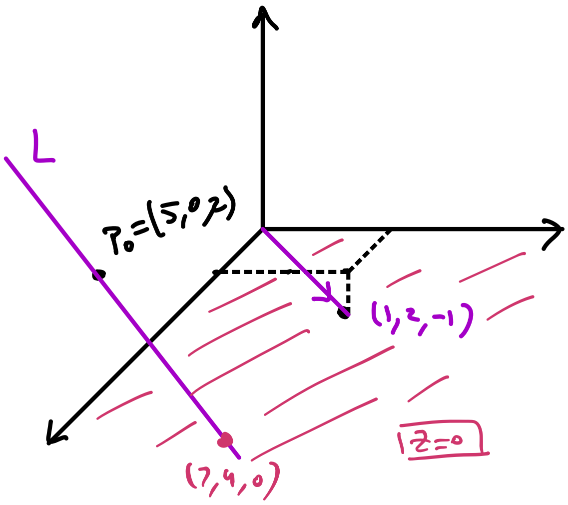
<u>Notice</u>: This is similar to how we write solutions to 2 linear equations in 3 vars (1 free param)

Another way to write $L$ is to eliminate $t$ from the 3 equations in (*)

- If $a, b, c \neq 0$ : $\boxed{\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}}$ Linear system with 3 eqns, 2 variables

Sometimes in the literature this is called the "symmetric form of the eqns of a line in $\mathbb{R}^3$. This way of writing it realizes $L$ as an intersection of 2 planes.

Examples (1) Find the equation of the line $L$ which is parallel to $\begin{bmatrix}1\\2\\-1\end{bmatrix}$ & passes through $(5,0,2)$




$P_0 = (5,0,2)$ $\quad\quad \vec{v} = \begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}1\\2\\-1\end{bmatrix}$

- $L: \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}5\\0\\2\end{bmatrix} + t\begin{bmatrix}1\\2\\-1\end{bmatrix}$

- $\dfrac{x-5}{1} = \dfrac{y-0}{2} = \dfrac{z-2}{-1}$

(2) Find the intersection of $L$ with the $xy$-plane

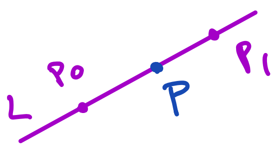
$xy$-plane has equation $z=0$, so we get one equation in $t$

$\begin{bmatrix}x\\y\\0\end{bmatrix} = \begin{bmatrix}5\\0\\2\end{bmatrix} + t\begin{bmatrix}1\\2\\-1\end{bmatrix} = \begin{bmatrix}5+t\\2t\\2-t\end{bmatrix}$ $\quad$ $3^{rd}$ comp gives $0 = 2-t$, or $t=2$

$P_t = (x,y,0) = (5+2, 2\cdot 2, 0) = (7,4,0)$

Alternatively, $\dfrac{x-5}{1} = \dfrac{y-0}{2} = \dfrac{0-2}{-1} = +2 \rightsquigarrow \begin{cases} x-5 = +2 \\ \frac{y}{2} = +2 \end{cases}$ $\quad x=7$, $y=4$

§3. Line Segments:




We are only interested in the points along $L$ in between $P_0$ & $P_1$

L: $\overrightarrow{OP} = \overrightarrow{OP_0} + t\,\overrightarrow{P_0P_1}$ ($t=0$ gives $P_0$, $t=1$ gives $P_1$)

This forces us to restrict $t$ to the interval $[0,1]$

Segment: $\boxed{\overrightarrow{P_0P} = t\,\overrightarrow{P_0P_1} \quad \text{for } 0 \le t \le 1}$

Special case: Midpoint $P$ between $P_0$ & $P_1$ comes from $\overrightarrow{P_0P} = \frac{1}{2}\overrightarrow{P_0P_1}$

$P_0 = (x_0, y_0, z_0)$

$P_1 = (x_1, y_1, z_1)$

$P = (x, y, z)$

$$\begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x_1-x_0 \\ y_1-y_0 \\ z_1-z_0 \end{bmatrix} \quad \text{gives} \quad \boxed{\begin{aligned} x &= \frac{x_0+x_1}{2} \\ y &= \frac{y_0+y_1}{2} \\ z &= \frac{z_0+z_1}{2} \end{aligned}}$$

## §4. Planes in $\mathbb{R}^3$

- A typical equation of a plane in $\mathbb{R}^3$ is

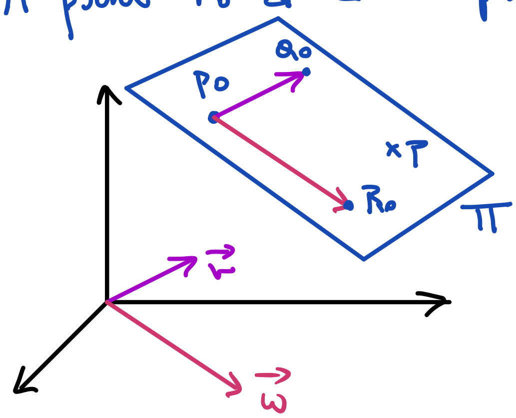
$$\boxed{ax + by + cz = d}$$

(1 linear eqn in 3 variables)

where either $a$, $b$ or $c \neq 0$

- Geometrically, a plane $\Pi$ in $\mathbb{R}^3$ is given in 2 possible ways:

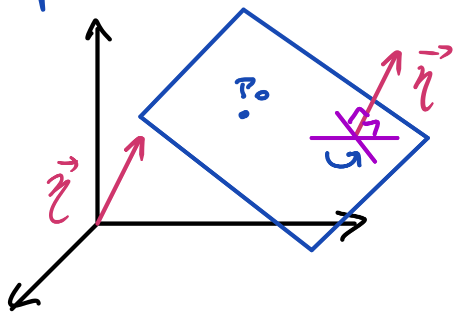
1. A point $P_0$ & 2 non-parallel direction $\vec{v}$ & $\vec{w}$.




Equivalently: 3 non collinear points $P_0, Q_0, R_0$ in $\mathbb{R}^3$

($\vec{v} = \overrightarrow{P_0Q_0}$, $\vec{w} = \overrightarrow{P_0R_0}$)

2. A point $P_0$ & a normal direction $\vec{n}$




- $\vec{n}$ orients the plane via the right-hand rule
- $\vec{n} \perp \vec{v}$ & $\vec{n} \perp \vec{w}$, so we can take $\vec{n} = \vec{v} \times \vec{w}$ or $\vec{w} \times \vec{v}$

**Vector equation:** $\overrightarrow{P_0P} \cdot \vec{n} = 0$

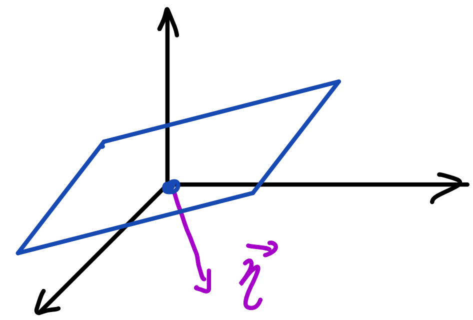
Explicitly: $P = (x, y, z)$, $P_0 = (x_0, y_0, z_0)$, $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightsquigarrow a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

**Examples** (1) $xy$-plane: it has equation $z=0$.

It is the plane passing through $(0,0,0)$ & perpendicular to $\vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(2) $2x+3y-z=0$ plane

- $P_0 = (0,0,0)$ (easy guess)
- $\vec{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ (from the equation)




(3) Find the equation of the plane passing through $(1,0,0)$, $(2,1,-1)$ & $(1\,1\,1)$

$\phantom{xx} = P_0 \qquad = Q_0 \qquad = R_0$

A: $\vec{v} = \overrightarrow{P_0Q_0} = \begin{bmatrix} 2-1 \\ 1-0 \\ -1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\vec{w} = \overrightarrow{P_0R_0} = \begin{bmatrix} 1-1 \\ 1-0 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\rightsquigarrow \vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = i \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - j \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} + k \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$

$= 2i - j + k = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Equation: $2(x-1) + (-1)(y-0) + 1(z-0) = 0$

$2x - y + z - 2 = 0 \rightsquigarrow \boxed{2x - y + z = 2}$

Check: $P_0$, $Q_0$ & $R_0$ satisfy the equation:

$2.1 - 0 + 0 = 2 \checkmark$ ; $2.2 - 1 + (-1) = 2 \checkmark$ , $2 - 1 + 1 = 2 \checkmark$