Lecture XIV: § 2.4: Planes in $\mathbb{R}^n$
§ 3.1, 3.2: $\mathbb{R}^n$ as a vector space

§1. Planes in $\mathbb{R}^3$

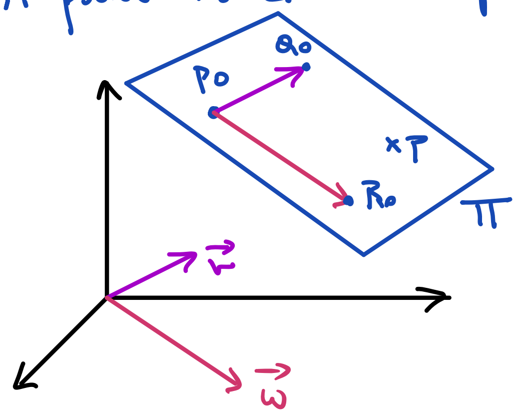
- A typical equation of a plane in $\mathbb{R}^3$ is

$$\boxed{ax + by + cz = d}$$ (1 linear eqn in 3 variables)

where either a, b or $c \neq 0$

- Geometrically, a plane Π in $\mathbb{R}^3$ is given in 2 possible ways:

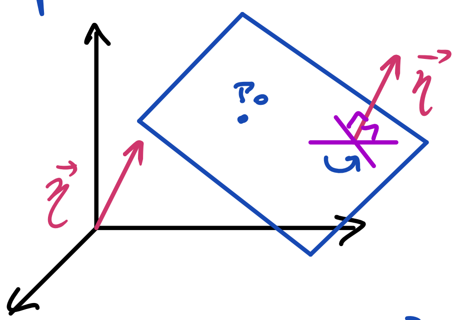
① A point P_0 & 2 non-parallel direction $\vec{v}$ & $\vec{w}$.



Equivalently: 3 non collinear points P_0, Q_0, R_0 in $\mathbb{R}^3$

$(\vec{v} = \overrightarrow{P_0Q_0}, \ \vec{w} = \overrightarrow{P_0R_0})$

② A point P_0 & a normal direction $\vec{n}$



- $\vec{n}$ orients the plane via the right-hand rule
- $\vec{n} \perp \vec{v}$ & $\vec{n} \perp \vec{w}$, so we can take $\vec{n} = \vec{v} \times \vec{w}$ or $\vec{w} \times \vec{v}$

Vector equation: $\overrightarrow{P_0P} \cdot \vec{n} = 0$

Explicitly: $P = (x, y, z)$
$P_0 = (x_0, y_0, z_0)$

$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightsquigarrow a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

Example: ① Describe the plane through $\overset{P_0}{(1,0,0)}$, $\overset{Q_0}{(0,1,0)}$ & $\overset{R_0}{(0,0,1)}$

$$\vec{n} = \overrightarrow{P_0Q_0} \times \overrightarrow{P_0R_0} = \det \begin{bmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = i - j(-1) + k = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

A: $1(x-1) + 1(y-0) + 1(z-0) = 0 \rightsquigarrow x + y + z = 1$

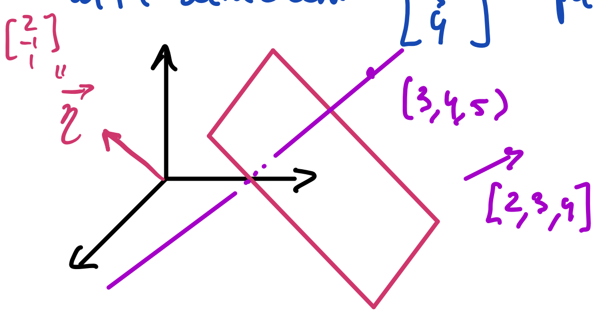
(2) Compute the intersection of the plane $2x - y + z = 2$ with the xy-plane

$$\begin{cases} 2x - y + z = 2 \\ z = 0 \end{cases} \rightsquigarrow 2x - y = 2 \quad \text{line in } \mathbb{R}^2$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 2x-2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad x \text{ in } \mathbb{R}.$$

Line through $(0,-2,0)$ with direction $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

(3) Compute the intersection of the plane $2x - y + z = 2$ with the line with direction $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ passing through $(3,4,5)$



Parametric eqn of the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3+2t \\ 4+3t \\ 5+4t \end{bmatrix}$$

Substitute these values in the equation of the plane.

$$2(3+2t) - (4+3t) + (5+4t) = 2$$

$$(6-4+5) + (4-3+4)t = 2$$

$$7 + 5t = 2 \quad \rightsquigarrow \boxed{t = -1}$$

So $P = (x,y,z) = (3-2, 4-3, 5-4) = (1,1,1)$.

OPTION 2

Solve the 3×3 system (2 eqns of line) + eqn of plane

$$\begin{cases} \frac{x-3}{2} = \frac{y-4}{3} \\ \frac{x-3}{2} = \frac{z-5}{4} \\ 2x - y + z = 2 \end{cases}$$

Definition 1: We say two planes are perpendicular or orthogonal to each other if their normal vectors are orthogonal.

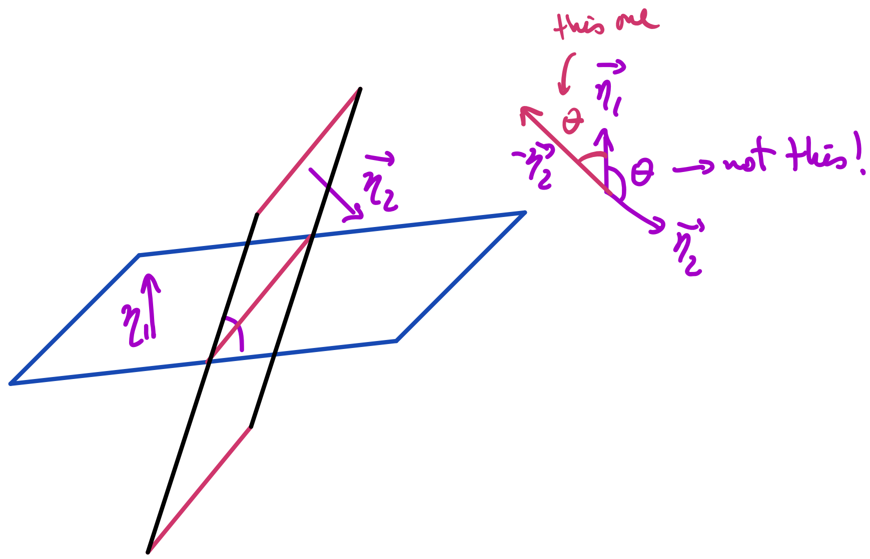
Definition 2: We say two planes are parallel if their normals are proportional

Example. $x - y + z = 0$ & $2x - 2y + 2z = 5$ are parallel

Reason: $\vec{n}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ & $\vec{n}_2 = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2\vec{n}_1$ so they are proportional

In general the angle between 2 planes is the acute angle between their normal vectors.

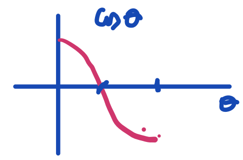
$0 \le \theta \le 90°$



<u>Example</u> $\quad x - y + z = 7 \quad$ plane 1 $\qquad \vec{n}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\qquad\qquad x + 4y + z = 2 \quad$ plane 2 $\qquad \vec{n}_2 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$

$$\vec{n}_1 \cdot \vec{n}_2 = 1 - 4 + 1 = -2 = \|\vec{n}_1\| \, \|\vec{n}_2\| \cos\theta$$



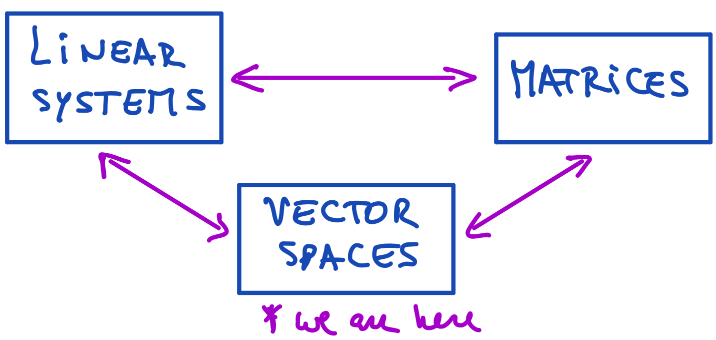
so $\quad 90^\circ < \theta < 180^\circ$

If we use $-\vec{n}_2$ we'll get the acute angle

$$\cos\theta = \frac{\vec{n}_1 \cdot (-\vec{n}_2)}{\|\vec{n}_1\| \, \|\vec{n}_2\|} = \frac{2}{\sqrt{3}\,\sqrt{18}} \qquad \text{so} \quad \theta = \cos^{-1}\left(\frac{2}{\sqrt{54}}\right)$$

<u>§2. Intro to $\mathbb{R}^n$ as a vector space</u>

Recall the original 3 components of this course



The Geometry of vectors in $\mathbb{R}^2$ & $\mathbb{R}^3$ (including lines & planes) was a warm-up for the algebra of vectors in n-space, which is our next topic

So far, we have seen 2 constructions:

① (Column) Vectors in $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4, \ldots$

② Solutions to <u>homogeneous</u> systems in $\mathbb{R}^n$ can be written in vector form as:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_{i_1}\,\vec{v_1} + x_{i_2}\,\vec{v_2} + \cdots + x_{i_s}\,\vec{v_s}$$

($x_{i_1}, x_{i_2}, \ldots, x_{i_s}$ are the independent variables of the system)

<u>Example</u>: $\begin{cases} x_1 + x_2 + 2x_4 + 3x_5 = 0 \\ x_3 + x_4 + x_5 = 0 \end{cases}$ $\quad B = \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$ $\quad x_2, x_4, x_5$ indep vars

dep

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_2 - 2x_4 - 3x_5 \\ x_2 \\ -x_4 - x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{=\vec{v_1}} + x_4 \underbrace{\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{=\vec{v_2}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{=\vec{v_3}} \qquad x_2, x_4, x_5 \text{ free}$$

<u>Q</u>: What do these 2 constructions have in common?

<u>A</u>:
- $\vec{0}$ lies in the space
- Add vectors / solutions gives a new vector / solution
- Scalar Multiplication preserves vectors / space of solutions

} 3 basic properties for vector spaces & subspaces

<u>§ 3. Defining Properties for $\mathbb{R}^n$</u> (same properties for abstract vector sp.)

<u>Theorem 1</u>: Write $\mathbb{V} = \mathbb{R}^n$. For $\vec{x}, \vec{y}, \vec{z}$ in $\mathbb{V}$, a, b scalars we have:

① <u>Closure Properties</u> (C1) If $\vec{x}, \vec{y}$ in $\mathbb{V}$, then $\vec{x} + \vec{y}$ in $\mathbb{V}$

(C2) If $\vec{x}$ in $\mathbb{V}$, then $a\vec{x}$ in $\mathbb{V}$ for all scalars a.

② <u>Addition Properties</u>: (A1) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (Commutative)

(A2) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ (Associative)

[Neutral Elem] (A3) $\vec{0}$ satisfies $\vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$ for all $\vec{x}$ in $\mathbb{V}$

[Additive Inverse] (A4) Given $\vec{x}$ in $\mathbb{V}$ we can find "$-\vec{x}$" in $\mathbb{V}$ satisfying $\vec{x} + (\text{"}-\vec{x}\text{"}) = \vec{0}$ (here "$-\vec{x}$" $= (-1)\vec{x}$)

③ Scalar Multiplication Properties:

(M1) $a(b\vec{x}) = (ab)\,\vec{x}$ (Associative)

(M2) $a(\vec{x}+\vec{y}) = a\vec{x} + a\vec{y}$ (Distributive I)

(M3) $(a+b)\,\vec{x} = a\vec{x} + b\vec{x}$ (———— II)

(M4) $1\,\vec{x} = \vec{x}$ for all $\vec{x}$

Note. (A4) follows from (C2) + $0\cdot\vec{x} = \vec{0}$.

§2. Subspaces of $\mathbb{R}^n$

Definition: A subspace S of $\mathbb{R}^n$ is a subset S of $\mathbb{R}^n$ that satisfies:

(S1) $\vec{0}$ is in S

(S2) for any $\vec{u}, \vec{v}$ in S, $\vec{u}+\vec{v}$ is in S

(S3) for any scalar a and vector $\vec{v}$ in S, $a\vec{v}$ is in S.

Examples: (1) $S = \{\vec{0}\}$ is a subspace of $\mathbb{R}^n$

(2) $S = \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$

(3) Subspaces of $\mathbb{R}^2$ are:
- $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ (dim 0)
- Lines through the origin (dim 1)
- $\mathbb{R}^2$ (dim 2)

(4) Subspaces of $\mathbb{R}^3$ are:
- $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$ (dim 0)
- Lines through the origin (dim 1)
- Planes ———— (dim 2)
- $\mathbb{R}^3$ (dim 3)