Lecture XV: §3.2 (II) Subspaces of $\mathbb{R}^n$
§3.3 Examples of subspaces

Last Time: • We introduce a vector space structure on $\mathbb{R}^n$ via 10 properties / axioms

• We ______ subspaces of $\mathbb{R}^n$ & gave examples

§1. Subspaces of $\mathbb{R}^n$:

Definition: A subspace $\mathbb{V}$ of $\mathbb{R}^n$ is a subset S of $\mathbb{R}^n$ that satisfies:

(S1) $\vec{0}$ is in $\mathbb{V}$

(S2) for any $\vec{u}, \vec{v}$ in $\mathbb{V}$, $\vec{u}+\vec{v}$ is in $\mathbb{V}$

(S3) for any scalar a and vector $\vec{v}$ in $\mathbb{V}$, $a\vec{v}$ is in $\mathbb{V}$

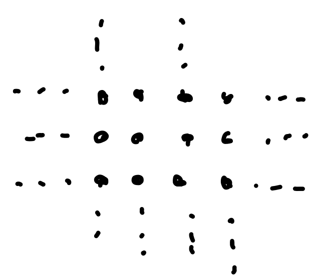
Non-examples: (1) $\mathbb{V} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x+y-z=4 \right\}$ is NOT a subspace of $\mathbb{R}^3$ because (S1) fails $(2\cdot 0+0-0=0\neq 4)$

(2) $\mathbb{V} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \text{ are integers} \right\}$ is NOT a subspace of $\mathbb{R}^2$

• $\vec{0}$ in S

• $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{V}$, then $\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}$ in $\mathbb{V}$

(sum of integers is an integer)

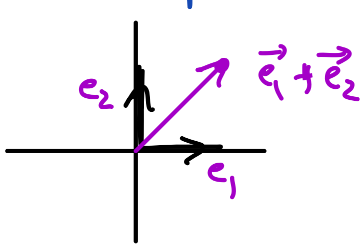


• (S3) fails $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in $\mathbb{V}$, $a=\frac{1}{2}$ means $\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ is not in $\mathbb{V}$

(3) Union of 2 different lines through $(0,0)$ is not a subspace

Ex: $\mathbb{V} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \text{either } x=0 \text{ or } y=0 \right\} = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} : y \text{ in } \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} ; x \text{ in } \mathbb{R} \right\}$

(S2) fails $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in $\mathbb{V}$ but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ not in $\mathbb{V}$.



Next: 2 meta examples

§2. Meta Example 1: Solutions to homog systems in n variables.

Theorem 1: Solutions to a homogeneous system of m equations in n variables form a subspace of $\mathbb{R}^n$.

Why? Write a system $\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases}$ as $A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ for $A = (a_{ij})_{i,j}$

We need to check (S1), (S2) & (S3).

(S1) $\vec{0}$ in $\mathbb{R}^n$ is a solution because $\underset{m\times n}{A} \cdot \underset{n \times 1}{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}} = \underset{m\times 1}{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}$

(S2) Pick $\vec{u}$ & $\vec{v}$ solutions, ie $A\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Then $A(\vec{u}+\vec{v}) \underset{\text{Distributive}}{=} A\vec{u} + A\vec{v} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ so $\vec{u}+\vec{v}$ is also a solution.

(S3) Pick a solution $\vec{u}$ & a scalar a, so $A\vec{u} = \vec{0}$.

Then, $A(a\vec{u}) \underset{\text{scalars jump}}{=} a(A\vec{u}) = a \cdot \vec{0} = \vec{0}$ so $a\vec{u}$ is a solution.

Another name for this subspace = the Null Space of the matrix $\underset{m\times n}{A}$.

$$\boxed{\mathcal{N}(A)} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^n : A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^m \right\}$$

= Solutions to the homog system with coeff matrix A.

§2. Meta Example 2: the span of a subset of vectors

Assume we are given m vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$ & consider the following set

$$\mathbb{V} = \text{all possible linear combinations of } \vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$$
$$= \{ a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m \text{ where } a_1, a_2, \dots, a_m \text{ are real numbers} \}$$
$$= Sp(\vec{v}_1, \dots, \vec{v}_m)$$

We call $\mathbb{V}$ the linear span of $\{\vec{v}_1, \dots, \vec{v}_m\}$

Theorem 2: $Sp(\vec{v}_1, \dots, \vec{v}_m)$ is a subspace of $\mathbb{R}^n$.

Why? We need to check (S1), (S2) & (S3).

(S1) $\vec{0}$ in $Sp(\vec{v}_1, \dots, \vec{v}_m)$ because $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \dots + 0\vec{v}_m$

(take $a_1 = a_2 = \dots = a_m = 0$)

(S2) Pick $\vec{u}$, $\vec{w}$ in $Sp(\vec{v}_1, \dots, \vec{v}_m)$, so

$$\begin{array}{rll} & \vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m & \text{for some } a_1, \dots, a_m \text{ scalars} \\ + & \vec{w} = b_1\vec{v}_1 + b_2\vec{v}_2 + \dots + b_m\vec{v}_m & \text{——— } b_1, \dots, b_m \text{ ———} \\ \hline \end{array}$$

$$\vec{u} + \vec{w} = (a_1\vec{v}_1 + \dots + a_m\vec{v}_m) + (b_1\vec{v}_1 + \dots + b_m\vec{v}_m)$$
$$\underset{\text{regroup}}{=} (a_1 + b_1)\vec{v}_1 + \dots + (a_m + b_m)\vec{v}_m \quad \text{is in } Sp(\vec{v}_1, \dots, \vec{v}_m)$$

(S3) Pick $\vec{w}$ in $Sp(\vec{v}_1, \dots, \vec{v}_m)$ & a = scalar.

Then $\vec{w} = b_1\vec{v}_1 + \dots + b_m\vec{v}_m \rightsquigarrow a\vec{w} = a(b_1\vec{v}_1 + \dots + b_m\vec{v}_m)$
$$= (ab_1)\vec{v}_1 + \dots + (ab_m)\vec{v}_m$$

confirms $a\vec{w}$ in $Sp(\vec{v}_1, \dots, \vec{v}_m)$

Ex: $\mathbb{V}$ = all linear comb of $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ in $\mathbb{R}^3$

W = plane with directions $\begin{bmatrix}1\\-1\\2\end{bmatrix}$ & $\begin{bmatrix}2\\0\\3\end{bmatrix}$ through $(0,0,0)$

Equation for W? $\vec{n} = \begin{bmatrix}1\\-1\\2\end{bmatrix} \times \begin{bmatrix}2\\0\\3\end{bmatrix} = \det\begin{pmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & 0 & 3\end{pmatrix} = \begin{bmatrix}-3\\1\\2\end{bmatrix}$

So equation is $-3x+y+2z=0$

Obs: $W = \mathcal{N}([-3,1,2]) = \left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} : [-3\ 1\ 2]\begin{bmatrix}x\\y\\z\end{bmatrix} = 0 \right\}$

This is ALWAYS TRUE! $\mathcal{N}(A)$ is a linear span of vectors (as many as $\#\text{Cols}\,A - \text{rank}\,A$).

Example Find vectors spanning $\mathcal{N}\left(\overset{=A}{\begin{bmatrix}1 & 1 & 0 & 2\\0 & 0 & 1 & 3\end{bmatrix}}\right)$

$$[A|\begin{smallmatrix}0\\0\end{smallmatrix}] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0\end{array}\right]$$

x_1, x_3 dependent
x_2, x_4 independent

$$\begin{cases} x_1 + x_2 + 2x_4 = 0 \\ x_3 + 3x_4 = 0 \end{cases}$$

We write the solutions in vector form, using x_2 & x_4 as parameters.

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-x_2-2x_4\\x_2\\-3x_4\\x_4\end{bmatrix} = \begin{bmatrix}-x_2\\x_2\\0\\0\end{bmatrix} + \begin{bmatrix}-2x_4\\0\\-3x_4\\x_4\end{bmatrix} = \underline{x_2}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + \underline{x_4}\begin{bmatrix}-2\\0\\-3\\1\end{bmatrix}$$

free

Conclusion: $\mathcal{N}(A) = \text{Sp}\left(\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\-3\\1\end{bmatrix}\right)$

§3 More examples:

① Column Space of an $m\times n$ matrix A. (subspace of $\mathbb{R}^m$)

Definition: The range or column space of A is

$$R(A) = \text{Sp}(\text{Col}_1 A, \ldots, \text{Col}_n A)$$

By construction it is a subspace of $\mathbb{R}^m$ (each column has m entries)

② Row Space of an $m\times n$ matrix A. (subspace of $\mathbb{R}^m$)

Definition: Row Space of A is $\text{RowSp}(A)$ = Span (row vectors of A viewed as col. vectors)

Example: $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$$R(A) = Sp\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right) = Sp\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right)$$

ignore repeated vector

Any vector $\begin{bmatrix}x\\y\end{bmatrix}$ in $\mathbb{R}^2$ is in $R(A)$. Indeed,

$$\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} + 0\begin{bmatrix}2\\3\end{bmatrix}$$

so $R(A) = \mathbb{R}^2 = Sp\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$

$$RowSp(A) = Sp\left(\begin{bmatrix}1\\1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\0\\1\\3\end{bmatrix}\right) = R\left(\begin{bmatrix}1 & 0\\1 & 0\\0 & 1\\2 & 3\end{bmatrix}\right) = R(A^T)$$

This is true in general! $\boxed{RowSp(A) = R(A^T)}$

So we can just focus on studying Ranges of matrices. The same ideas will translate to row spaces.