Lecture XVI: §3.3 (II) Examples of subspaces

Recall: Last Time we introduced 2 meta examples of subspaces

① $\underset{m\times n}{\mathcal{N}(A)} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^n \text{ with } A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^m \right\}$

= solutions to the homogeneous linear system in n variables with coefficient matrix A.

② $Sp(\vec{v}_1, \ldots, \vec{v}_s)$ = all linear combinations of the vectors $\vec{v}_1, \ldots, \vec{v}_s$ in $\mathbb{R}^n$

vectors in $\mathbb{R}^n$ $= \{ a_1\vec{v}_1 + \cdots + a_s\vec{v}_s : a_1, \ldots, a_s \text{ are scalars} \}$

- Two examples of spans involving matrices:

  Range (A) = Column Span (A) = $Sp(Col_1 A, \ldots, Col_n A)$ subspace of $\mathbb{R}^n$

  Row Span (A) = Range (A^T) (subspace of $\mathbb{R}^m$)

THEOREM: $\mathcal{N}(A)$ & $Sp(\vec{v}_1, \ldots, \vec{v}_s)$ are subspaces of $\mathbb{R}^n$

- By writing the solutions to $A\vec{x} = \vec{0}$ in vector form we can realize $\mathcal{N}(A)$ as a span of vectors (# vectors = # independent variables)

Q: Can we always realize spans as null-spaces of matrices? Equivalently, can we find equations for spans?

A: YES! We will see how today!

§1. Equations for spans:

Example 1: $A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 4 & 0 \\ 1 & 1 & 5 & -3 \end{bmatrix} \rightsquigarrow R(A) = Sp\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \right)$

Q: What is this subspace?

A A vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ is in $R(A)$ if we can find scalars x_1, x_2, x_3, x_4

with $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1 Col_1 A + x_2 Col_2 A + x_3 Col_3 A + x_4 Col_4 A = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

To "eliminate" $x_1,\dots,x_4$ from this description, we need to find conditions on y_1,y_2,y_3 that ensures the system $A\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$ can be solved.

We'll see how to do this in this example:

$$[A\,|\,\vec{y}\,] = \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1 \\ 2 & -1 & 4 & 0 & y_2 \\ 1 & 1 & 5 & -3 & y_3 \end{array}\right] \xrightarrow[R_3\to R_3-R_1]{R_2\to R_2-2R_1} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1 \\ 0 & 1 & 2 & -2 & y_2-2y_1 \\ 0 & 2 & 4 & -4 & y_3-y_1 \end{array}\right]$$

$$\xrightarrow[R_3\to R_3-2R_2]{} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1 \\ 0 & 1 & 2 & -2 & y_2-2y_1 \\ 0 & 0 & 0 & 0 & y_3-y_1-2(y_2-2y_1) \end{array}\right]$$

The system has solutions if and only if the last row is $[0\,0\,0\,0\,0]$

This says $y_3-y_1-2(y_2-2y_1) = y_3-y_1-2y_2+4y_1 = \boxed{3y_1-2y_2+y_3=0}$

Conclusion: $\mathcal{R}(A)$ is the plane in $\mathbb{R}^3$ with eqn $3y_1-2y_2+y_3=0$

Also: $\mathcal{R}(A) = \mathcal{N}([3\ \ -2\ \ 1])$

Example 2: $A = \begin{bmatrix} 1 & -4 & 2 \\ -1 & 5 & 2 \\ 2 & -8 & 5 \end{bmatrix}$ Describe $\mathcal{R}(A)$ via equations.

Again $\mathcal{R}(A)$ consists of all column vectors $\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ so that $A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ is consistent.

To check this, we use Gauss-Jordan elimination:

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & b_1 \\ -1 & 5 & 2 & b_2 \\ 2 & -8 & 5 & b_3 \end{array}\right] \xrightarrow[R_3\to R_3-2R_1]{R_2\to R_2+R_1} \left[\begin{array}{ccc|c} 1 & -4 & 2 & b_1 \\ 0 & 1 & 4 & b_2+b_1 \\ 0 & 0 & 1 & b_3-2b_1 \end{array}\right]$$

The system is ALWAYS consistent, with NO restrictions on b_1, b_2, b_3.

Conclusion: $\mathcal{R}(A) = \mathbb{R}^3$.

In general: Range of A is either $\mathbb{R}^n$ or it is the null-space of a matrix.

Method: $\left[A \,\middle|\, \begin{matrix} b_1 \\ \vdots \\ b_n \end{matrix} \right] \xrightarrow[\text{row red}]{} \left[\begin{array}{c|c} A' & \text{linear expressions in } b_1, \dots, b_n \\ \hline 0 \cdots 0 & \boxed{\text{linear expressions in } b_1, \dots, b_n} \\ 0 \cdots 0 & \end{array}\right]$

with A' in reduced ech form and no all zero rows.

The equations for $R(A)$ are obtain by equating to 0 all the expressions in the lower-right corner $\boxed{}$. (As in Example 2, there could be no conditions!)

§ 2. Spanning sets

Q: What happens to the row space of a matrix under elementary row operations?

A Same row space! (we'll see later why)

Advantage: We can use row operations to find a better set of generators for Row(A) & in general for $Sp(\vec{v}_1, \dots, \vec{v}_m)$ in $\mathbb{R}^n$.

$$A = \begin{bmatrix} \vec{v}_1^{\,t} \\ \vdots \\ \vec{v}_m^{\,t} \end{bmatrix} \xrightarrow{\text{Gauss Jordan}} C = \begin{bmatrix} \vec{w}_1^{\,t} \\ \vdots \\ \vec{w}_r^{\,t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{says } Rows(A) = Rows(C)$$

In particular: $Sp(\vec{v}_1, \dots, \vec{v}_m) = Sp(\vec{w}_1, \dots, \vec{w}_r)$ (can ignore $\vec{0}$'s in the span)

Example: $W = Sp\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix} \right)$ (4 generators)

Step 1: Write vectors as the Rows of a matrix A $\quad A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \\ -1 & -1 & -4 \end{bmatrix}$ 4x3

Step 2: Do Gauss-Jordan to find $C = REF(A)$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \\ -1 & -1 & -4 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_2 \\ R_4 \to R_1 + R_4}]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_2 \\ R_4 \to R_4 + R_2 \\ R_2 \to -R_2}]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 2R_2]{} \underbrace{\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{REF}$$

Step 3: Pick non-zero rows of C & write them as column vectors.

$$W = Sp\left(\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right)$$ 2 generators instead of 4

Q: Can we do better?

A NO! 2 is the minimal number of generators we need (vectors are l.i)

These generators have an additional advantage: we can easily find equations defining W

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in W has the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ for some a, b

But we see that $a = x$, $b = y$ & so $z = 7a - 3b = 7x - 3y$

$\rightsquigarrow$ Equation for W is $7x - 3y - z = 0$.