Lecture XVII: §3.4 Bases for subspaces

Last time: • we discussed how to find equations for subspaces given by spanning sets
• How to find "optimal" spanning sets for $Sp(\{\vec{v}_1, \ldots, \vec{v}_m\})$ in $\mathbb{R}^n$.

INPUT: $\{\vec{v}_1, \ldots, \vec{v}_m\}$ vectors in $\mathbb{R}^n$

OUTPUT: $\{\vec{w}_1, \ldots, \vec{w}_s\}$ minimal spanning set for $W = Sp(\vec{v}_1, \ldots, \vec{v}_m\}$

STEP 1: Write $A = \begin{bmatrix} \vec{v}_1^{\,t} \\ \vdots \\ \vec{v}_m^{\,t} \end{bmatrix}$ $m\times n$ matrix with $RowSp(A) = W$.

STEP 2: Write $REF(A) = \begin{bmatrix} \vec{w}_1^{\,t} \\ \vdots \\ \vec{w}_s^{\,t} \\ 0 \\ 0 \end{bmatrix}$ } non-zero rows of REF(A)

Return: $\{\vec{w}_1, \ldots, \vec{w}_s\}$

This algorithm works because of the following statement:

Theorem: Row equivalent matrices $(A \underset{row}{\sim} A')$ have equal row space.

Why? It is enough to check that each elementary row operation preserves row spaces.

$$\overset{m\times n}{A} = A_1 \xrightarrow{\text{Elem}} A_2 \xrightarrow{\text{Elem}} A_3 \xrightarrow{\text{Elem}} \cdots\cdots \xrightarrow{\text{Elem}} A_k = \overset{m\times n}{A'}$$

Then: $RowSp(A) = RowSp(A_2) = \cdots = RowSp(A_k) = RowSp(A')$

(E1) [Swap] ✓ Swap 2 rows reorders the vectors spanning the row space.

(E2) [Scale] Multiply a row (say R1) by a non-zero number $b \neq 0$.

$\vec{x}$ in $Sp(\vec{R}_1^{\,t}, \ldots, \vec{R}_m^{\,t}) \overset{?}{=} Sp(b\vec{R}_1^{\,t}, \vec{R}_2^{\,t}, \ldots, \vec{R}_m^{\,t})$

$$x = \underbrace{a_1}\vec{R}_1^{\,t} + \cdots + \underbrace{a_m}\vec{R}_m^{\,t} = \underbrace{\tfrac{a_1}{b}}(b\vec{R}_1^{\,t}) + \underbrace{a_2}\vec{R}_2^{\,t} + \cdots + \underbrace{a_m}\vec{R}_m^{\,t}. \quad ✓$$

we see how to modify scalars to confirm $\underline{x}$ is in both spaces.

(E3) [Combine] Say we replace $R_1 \to R_1 + bR_2$ for b = scalar

$$Sp(\vec{R}_1^t, \ldots, \vec{R}_m^t) \stackrel{?}{=} Sp(\vec{R}_1^t + b\vec{R}_2^t, \vec{R}_2^t, \ldots, \vec{R}_m^t)$$

$$\vec{x} = \underbrace{a_1}\vec{R}_1^t + \cdots + \underbrace{a_m}\vec{R}_m^t = \underbrace{a_1}(\vec{R}_1^t + b\vec{R}_2^t) + \underbrace{(a_2 - a_1 b)}\vec{R}_2^t + \underbrace{a_3}\vec{R}_3^t + \cdots \underbrace{a_m}\vec{R}_m^t) \checkmark$$

we see how to modify scalars to confirm $\underline{x}$ is in both spaces.

Conclusion: (E1), (E2) & (E3) changes the vectors individually, but preserves their span.

Q. In which sense is the output minimal?

A. $\{\vec{w}_1, \ldots, \vec{w}_s\}$ are l.i (because of the staircase shape of REF(A)), so there are no relations among the vectors. In particular, we cannot remove any of them and span the same set. In this sense, the set is minimal

§1 Bases for subspaces I

Definition: A basis for a subspace $\mathbb{W}$ of $\mathbb{R}^n$ is a minimal spanning set B for $\mathbb{W}$. (removing a vector from the set B will no longer span $\mathbb{W}$)

Later on, we'll see that minimal spanning sets are always lin. indep.

The algorithm discussed above produces a basis for $\mathbb{W}$.

Example: $\mathbb{W} = Sp\left(\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\4\\5\end{bmatrix}, \begin{bmatrix}1\\0\\-3\end{bmatrix}\right)$ Find a basis for $\mathbb{W}$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 1 & 4 & 5 \\ 1 & 0 & -3 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 + 2R_2}]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 2R_2]{} \underbrace{\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{REF(A)}$$

Basis $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

Clearly l.i $\quad x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives $\begin{cases} x_1 = 0 \\ x_2 = 0 \\ -3x_1 + 2x_2 = 0 \end{cases}$

$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ is <u>not</u> in $Sp\left(\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}\right)$ & $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ is not in $Sp\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right)$

So $\mathbb{W} \neq Sp\left(\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}\right)$ & $\mathbb{W} \neq Sp\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right)$

Easy to find equations for $\mathbb{W}$ using B.

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underline{a_1} \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + \underline{a_2} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ gives $\begin{cases} x = a_1 \\ y = a_2 \\ z = -3a_1 + 2a_2 = -3x + 2y. \end{cases}$

(scalars to find)

So equation for $\mathbb{W}$ is $\boxed{-3x + 2y - z = 0}$.

<u>Observation</u>: The output B has nothing to do with the original spanning set of 4 vectors.

<u>Q</u> Can we produce a basis among the original vectors?

<u>A</u>: YES!

<u>Example</u>: $\mathbb{W} = Sp\left(\underset{\vec{v}_1}{\underset{\shortparallel}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}}, \underset{\vec{v}_2}{\underset{\shortparallel}{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}}, \underset{\vec{v}_3}{\underset{\shortparallel}{\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}}}, \underset{\vec{v}_4}{\underset{\shortparallel}{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}}} \right)$

- $\vec{v}_1$ is in $Sp\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \right) \rightsquigarrow$ can remove $\vec{v}_1$

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \stackrel{?}{=} a_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ -1 & 4 & 0 & 2 \\ 1 & 5 & -3 & 1 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_3]{} \left[\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ -1 & 4 & 0 & 2 \\ -1 & 1 & 1 & 1 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1}]{} \left[\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & 9 & -3 & 3 \\ 0 & 6 & -2 & 2 \end{array}\right]$$

$$\xrightarrow[R_2 \to \frac{R_2}{9}]{} \left[\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & 1 & -1/3 & 1/3 \\ 0 & 6 & -2 & 2 \end{array}\right] \xrightarrow[R_3 \to R_3 - 6R_2]{} \left[\begin{array}{ccc|c} 1 & 5 & -3 & 1 \\ 0 & 1 & -1/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 - 5R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -2/3 \\ 0 & 1 & -1/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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so $a_1 = -\frac{2}{3} + \frac{4}{3} a_3$
$a_2 = -\frac{1}{3} - a_3$ for any a_3 (eg $a_3 = 0$ gives $\vec{v_1} = \frac{-2}{3}\vec{v_2} + \left(\frac{-1}{3}\right)\vec{v_3}$)

Conclusion $Sp(\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}) = Sp(\vec{v_2}, \vec{v_3}, \vec{v_4})$

Why? $b_1 \vec{v_1} + b_2 \vec{v_2} + b_3 \vec{v_3} + b_4 \vec{v_4} = b_1\left(-\frac{2}{3}\vec{v_2} + \left(\frac{-1}{3}\right)\vec{v_3}\right) +$
$+ b_2\vec{v_2} + b_3\vec{v_3} + b_4\vec{v_4} = \left(-\frac{2}{3}b_1 + b_2\right)\vec{v_2} + \left(b_3 - \frac{1}{3}b_1\right)\vec{v_3} + b_4\vec{v_4}$
shows $Sp(\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}) = Sp(\vec{v_2}, \vec{v_3}, \vec{v_4})$

- $\vec{v_2}$ is in $Sp\left(\begin{bmatrix}1\\4\\5\end{bmatrix}, \begin{bmatrix}1\\0\\-3\end{bmatrix}\right)$ ⇝ we can remove $\vec{v_2}$

$$\begin{bmatrix}-1\\-1\\1\end{bmatrix} \stackrel{?}{=} a_1\begin{bmatrix}1\\4\\5\end{bmatrix} + a_2\begin{bmatrix}1\\0\\-3\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 1 & -1\\4 & 0 & -1\\5 & -3 & 1\end{array}\right] \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 4R_1} \left[\begin{array}{cc|c}1 & 1 & -1\\0 & -4 & -3\\0 & -8 & 6\end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{cc|c}1 & 1 & -1\\0 & -4 & -3\\0 & 0 & 0\end{array}\right] \xrightarrow{R_2 \to \frac{R_2}{-4}} \left[\begin{array}{cc|c}1 & 1 & -1\\0 & 1 & 3/4\\0 & 0 & 0\end{array}\right]$$

so $a_1 = -1 - a_2 = -1 - 3/4 = \frac{-7}{4}$ $\qquad \vec{v_2} = \frac{-7}{4}\vec{v_3} + \frac{3}{4}\vec{v_4}$
$a_2 = \frac{3}{4}$

Conclusion: $Sp(\vec{v_2}, \vec{v_3}, \vec{v_4}) = Sp(\vec{v_3}, \vec{v_4})$

Why? $a\vec{v_2} + b\vec{v_3} + c\vec{v_4} = a\left(-\frac{7}{4}\vec{v_3} + \frac{3}{4}\vec{v_4}\right) + b\vec{v_3} + c\vec{v_4}$
$= \left(\frac{-7}{4}a + b\right)\vec{v_3} + \left(\frac{3}{4}a + c\right)\vec{v_4}.$

Conclusion $W = Sp(\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}) = Sp(\vec{v_3}, \vec{v_4})$.

- $\{\vec{v_3}, \vec{v_4}\}$ are li, so the set is a minimal spanning set.

§1 Bases for subspaces II

The example shows a general procedure to extract a basis from a spanning set.

INPUT: A spanning set $S = \{\vec{v_1}, \ldots, \vec{v_p}\}$ for a subspace W of $\mathbb{R}^n$

OUTPUT: A subset T of S that is a basis for W.

STEP 1 Is S l.i.? $\to$ If YES, then output S
$\to$ If NO, find a nontrivial relation

say $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_p\vec{v}_p = \vec{0}$ & pick first (or any) $a_i \neq 0$

For example, say $a_1 \neq 0$. Then $\vec{v}_1 = -\frac{a_2}{a_1}\vec{v}_2 + \cdots + \left(-\frac{a_p}{a_1}\right)\vec{v}_p$

New $S = \{\vec{v}_2, \ldots, \vec{v}_p\}$

(In general new $S = \{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_p\} = S \setminus \{\vec{v}_i\}$)

STEP 2 Repeat Step 1 for New S, etc.

At some point we get a l.i. subset & this is our output.

Example. $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix} \right\}$

- S is l.d. Write $7\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \\ -7 \end{bmatrix} + (-2)\begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\vec{v}_1 = -\frac{3}{7}\vec{v}_2 + \frac{2}{7}\vec{v}_3$.

- New $S = \left\{ \begin{bmatrix} -1 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix} \right\}$ is l.i.

($a\vec{v}_2 + b\vec{v}_3 = \vec{0}$ gives $\begin{aligned} -a+2b &= 0 \\ 7b &= 0 \\ -7a &= 0 \end{aligned}$ so $a=b=0$ is the only solution)

So New S is a basis.

Observe: The algorithm gives a new characterization for basis!
(OUTPUT is always l.i.)

Proposition: A set B is a basis for W if
(1) B is a spanning set
& (2) B is l. indep.