Lecture XVIII: §3.4 Bases for subspaces
§3.5 Dimension of subspaces

Recall: A basis B for a subspace W of $\mathbb{R}^n$ is a set $\{w_1, \ldots, w_d\}$ spanning W & minimal with this property.

Equivalently: B is a basis if

(1) B spans W

(2) B is linearly independent

Obs: $\{\vec{0}\}$ is l.d, so the subspace $\{\vec{0}\}$ has no basis. It is the only subpace with no basis.

§1. Constructing bases:

- We saw two ways to build a basis for $W = Sp(\vec{v}_1, \ldots, \vec{v}_m)$

Method ①:

$$\underset{m\times n}{A} = \begin{bmatrix} \vec{v}_1^{\,t} \\ \vdots \\ \vec{v}_m^{\,t} \end{bmatrix} \xrightarrow[\text{Gauss-Jordan}]{} REF(A) = \begin{bmatrix} \vec{w}_1^{\,t} \\ \vdots \\ \vec{w}_d^{\,t} \\ \hline 0 \cdots 0 \\ \vdots \\ 0 \cdots 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} \text{nonzero rows} \\ \\ \end{matrix}$$

$B = \{\vec{w}_1, \ldots, \vec{w}_d\}$ is a basis for W

Method ②: Remove vectors from $S = \{\vec{v}_1, \ldots, \vec{v}_m\}$ one at a time using linear dependencies

Q: Is S li?

- If YES, then $B = S$

  If NO, then use a dependency to write some ω_i in terms of the others, & repeat the question for the new set $S' = \{\vec{v}_1 \cdots \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_m\} = S \smallsetminus \{v_i\}$.

Example: (1) $\left\{ \underset{=\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \underset{=\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \ldots, \underset{=\vec{e}_n}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}} \right\}$ is a basis for $W = \mathbb{R}^n$

called the standard basis

(2) $S = \left\{ \begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$ is not a basis for $\mathbb{R}^2$ (l.d.!)

A dependency relation is $\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

We can choose to remove any of the vectors. The other 2 are li, so they give a basis:

3 options : $\left\{ \begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \right\}$, $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$

no $\vec{v}_3$ no $\vec{v}_2$ no $\vec{v}_1$

Q: Can we do better than Method 2?

A YES, we can do this in 1 step using the following trick

Method ③:

- View $\mathbb{V} = \mathrm{Sp}(\vec{v}_1, \ldots, \vec{v}_p)$ as the Range of the matrix $\underset{n\times p}{A} = [\vec{v}_1 \cdots \vec{v}_p]$

- Basis $= \{ \vec{v}_i$: i is a column corresponding to a DEPENDENT variable of $A \begin{bmatrix}x_1\\ \vdots \\ x\end{bmatrix} = \vec{0} \}$

Example: $\mathbb{V} = \mathrm{Sp}\left(\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}3\\5\\6\end{bmatrix} \begin{bmatrix}-1\\-1\\-4\end{bmatrix} \right)$

$$\left[A \middle| \begin{matrix}0\\0\\0\end{matrix}\right] = \left[\begin{matrix}1&2&3&-1\\2&3&5&-1\\1&5&6&-4\end{matrix}\middle|\begin{matrix}0\\0\\0\end{matrix}\right] \xrightarrow[\substack{R_2\to R_2-2R_1\\R_3\to R_3-R_1}]{} \left[\begin{matrix}1&2&3&-1\\0&-1&-1&1\\0&3&3&-3\end{matrix}\middle|\begin{matrix}0\\0\\0\end{matrix}\right] \xrightarrow[\substack{R_3\to R_3+3R_2\\R_2\to -R_2}]{} \overset{EF}{\left[\begin{matrix}1&2&3&-1\\0&1&1&1\\0&0&0&0\end{matrix}\middle|\begin{matrix}0\\0\\0\end{matrix}\right]}$$

$$\xrightarrow[R_1\to R_1-2R_2]{} \left[\begin{matrix}1&0&1&-3\\0&1&1&1\\0&0&0&0\end{matrix}\middle|\begin{matrix}0\\0\\0\end{matrix}\right] \quad \text{REF}$$

x_1, x_2 dependent $\rightsquigarrow$ Basis $= \left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\3\\5\end{bmatrix} \right\}$

Q Why does this work? ← can use REF(A) instead of A.

$$\mathrm{Null}(A) = \mathrm{Null}(\mathrm{REF}(A)) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : \begin{array}{l} x_1 + x_3 - 3x_4 = 0 \\ x_2 + x_3 + x_4 = 0 \end{array} \right\}$$

$$\begin{cases} x_1 = -x_3 + 3x_4 \\ x_2 = -x_3 - x_4 \end{cases} \rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 + 3x_4 \\ -x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

This gives 2 dependency relations among $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

$$\vec{0} = A \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = -\vec{v}_1 - \vec{v}_2 + \vec{v}_3 \quad \& \quad \vec{0} = A \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix} = 3\vec{v}_1 - \vec{v}_2 + \vec{v}_4$$

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 \qquad\qquad \vec{v}_4 = -3\vec{v}_1 + \vec{v}_2$$

So $\mathrm{Sp}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \mathrm{Sp}(\vec{v}_1, \vec{v}_2)$ & $\{\vec{v}_1, \vec{v}_2\}$ is li

so it is a basis for $\mathbb{W}$.

§2. Some properties of bases:

Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d\}$ be a basis for $\mathbb{W}$ (subspace of $\mathbb{R}^n$)

Property 1: Uniqueness of expression

For $\vec{w}$ in $\mathbb{W}$, we can find uniquely determined numbers.

$a_1, a_2, \dots, a_p$ such that $\vec{w} = a_1\vec{v}_1 + \dots + a_d\vec{v}_d$

Call them the coordinates of $\vec{w}$ with respect to the basis $\mathcal{B}$ & write it as $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix}$ (vector in $\mathbb{R}^d$)

Why? • We have such expression because $\mathcal{B}$ spans $\mathbb{W}$.

• If we have 2 such expressions $b_1\vec{v}_1 + \dots + b_d\vec{v}_d = a_1\vec{v}_1 + \dots + a_d\vec{v}_d$

for $\vec{w}$, then taking their difference, we get

$$(a_1-b_1)\vec{v}_1 + \cdots + (a_d-b_d)\vec{v}_d = \vec{0}$$

Since $\vec{v}_1, \ldots, \vec{v}_d$ are l.i., we must have $a_1-b_1 = a_2-b_2 = \cdots = a_d-b_d = 0$, in other words, $a_1=b_1$, $a_2=b_2$, $\ldots$, $a_d=b_d$. This shows the expression is unique!

Property 2: Size of a basis is fixed

Fix a set $S=\{\vec{w}_1, \ldots, \vec{w}_m\}$ of vectors in W with $m>d$. Then S is linearly dependent.

(Compare: $W=\mathbb{R}^n$ $B=\{\vec{e}_1, \ldots, \vec{e}_n\}$ $(d=n)$ standard basis, any set with $m>n$ elements is always linearly dependent

($\underset{n\times m}{A}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ with $A=\begin{bmatrix} \vec{w}_1, \ldots, \vec{w}_m \end{bmatrix}$ is a homogeneous system with n equations & unknowns. But $m>n$ so the system has ∞-many solutions in particular $\{\vec{w}_1, \ldots, \vec{w}_m\}$ are l.d.).

Why? Write each $\vec{w}_1, \ldots, \vec{w}_m$ as a linear combination of $\vec{v}_1, \ldots, \vec{v}_d$:
(using Property 1)

$$\vec{w}_1 = a_{11}\vec{v}_1 + \cdots + a_{1d}\vec{v}_d$$
$$\vec{w}_2 = a_{21}\vec{v}_1 + \cdots + a_{2d}\vec{v}_d$$
$$\vdots$$
$$\vec{w} = a_{m1}\vec{v}_1 + \cdots + a_{md}\vec{v}_d$$

$\rightsquigarrow$ Build $\underset{m\times d}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{md} \end{bmatrix}$

Write a relation $x_1\vec{w}_1 + \cdots + x_m\vec{w}_m = \vec{0}$.

Substituting expressions of $\vec{w}$'s in terms of $\vec{v}$s gives us a homogeneous system

$$(a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m)\vec{v}_1 + (a_{12}x_1 + \cdots + a_{m2}x_m)\vec{v}_2 + \cdots + (a_{1d}x_1 + a_{2d}x_2 + \cdots + a_{md}x_m)\vec{v}_d = \vec{0}$$

(coefficients are the entries of the row vector $[x_1 \cdots x_m]\, A = \left(A^T \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}\right)^T$)

But $\{\vec{v}_1, \ldots, \vec{v}_d\}$ are li, so we get a system:

$$\begin{aligned} a_{11} x_1 + a_{21} x_2 + \cdots + a_{m1} x_m &= 0 \\ a_{12} x_1 + a_{22} x_2 + \cdots + a_{m2} x_m &= 0 \\ &\vdots \\ a_{1d} x_1 + a_{2d} x_2 + \cdots + a_{md} x_m &= 0 \end{aligned} \quad = \underbrace{A^T}_{d \times m} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \vec{0}$$

Since $d < m$ (# eqns < # unknowns) so the system has ∞-many solns $(x_1, \ldots, x_m)$. This says $\{\vec{w}_1, \ldots, \vec{w}_m\}$ is l.dep.

§3. Dimension of a subspace $\mathbb{W}$.

As a consequence of property 2, we get.

THEOREM: If $B_1 = \{\vec{v}_1, \ldots, \vec{v}_d\}$, $B_2 = \{\vec{w}_1, \ldots, \vec{w}_m\}$ are bases for $\mathbb{W}$, then $d = m$

We call this number the dimension of $\mathbb{W}$. Write $\dim \mathbb{W} = d$.

Why? If $d < q$ then B_2 is l.d (use B_1 is a basis), which cannot happen.

If $q < d$ ——— B_1 ——— (use B_2 ———), ———————————

Note: A subspace can have many bases, but all of them have the same number of vectors.

Consequence: Fix $\mathbb{W} \neq \{\vec{0}\}$ a subspace of $\mathbb{R}^n$ with $\dim \mathbb{W} = d$. Then:

1. Any set of $d+1$ or more vectors in $\mathbb{W}$ is lin. dep.
2. ——————— $d-1$ ——————————— cannot span $\mathbb{W}$
3. ——————— d lin. indep vectors in $\mathbb{W}$ is a basis for $\mathbb{W}$.
4. ——————— d vectors that spans $\mathbb{W}$ is a basis for $\mathbb{W}$.

<u>Example:</u> $W = Sp\left(\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}3\\5\\6\end{bmatrix}, \begin{bmatrix}-1\\-1\\-4\end{bmatrix}\right)$

"$\vec{v}_1$" "$\vec{v}_2$" "$\vec{v}_3$" "$\vec{v}_4$"

Last Time we computed a basis for W $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\7\end{bmatrix}, \begin{bmatrix}0\\1\\-3\end{bmatrix}\right\}$ so dim $W = 2$

- Consequence ① confirms $\{\vec{v}_1, \dots, \vec{v}_4\}$ is l d
- ______ ② $\{\vec{v}_1\}$ cannot span W. Same if we pick any individual vector $\{\vec{v}_2\}$, $\{\vec{v}_3\}$, $\{\vec{v}_4\}$
- Consequence ③ $\{\vec{v}_1, \vec{v}_2\}$ is l i

$$\left[\begin{array}{cc|c}1&2&0\\2&3&0\\1&5\end{array}\right] \xrightarrow[R_3\to R_3-R_1]{R_2\to R_2-2R_1} \begin{bmatrix}1&2\\0&-1\\0&3\end{bmatrix} \xrightarrow[R_2\to R_2/-1]{R_3\to R_3+3R_2} \begin{bmatrix}1&2\\0&1\\0&0\end{bmatrix}$$

So it is a basis for W.

Similarly $\{\vec{v}_1, \vec{v}_3\}$, $\{\vec{v}_1, \vec{v}_4\}$, $\{\vec{v}_2, \vec{v}_3\}$, $\{\vec{v}_2, \vec{v}_4\}$, $\{\vec{v}_3, \vec{v}_4\}$ are li of size 2, so they are bases for W.