Lecture XX: §3.6 Gram-Schmidt Algorithm

Recall: Fix usual dot product in $\mathbb{R}^n$: $\vec{u}\cdot\vec{v} = \underset{1\times n}{\vec{u}^T}\,\underset{n\times 1}{\vec{v}}$ (matrix mult.)

$\vec{u}\perp\vec{v}$ (orthogonal) if $\vec{u}\cdot\vec{v}=0$

- $S=\{\vec{v}_1,\ldots,\vec{v}_m\}$ is an orthogonal set if $\vec{v}$'s are mutually orthogonal
- S is orthonormal if orthogonal + $\|\vec{v}_i\|=\sqrt{\vec{v}_i\cdot\vec{v}_i}=1$ for all $i=1,\ldots,m$

We have orthogonal / orthonormal bases for subspaces $\mathbb{W}$ of $\mathbb{R}^n$ with $\mathbb{W}\neq\{\vec{0}\}$

Key Property: If S is an orthogonal set in $\mathbb{R}^n$ & $\vec{0}$ is not in S, then S is linearly independent.

§3 Coordinates with respect to orthogonal bases.

Example: $S=\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}-1\\-1\\1\end{bmatrix},\begin{bmatrix}5\\-4\\1\end{bmatrix}\right\}$ subset of vectors in $\mathbb{R}^3$

- $\vec{v}_1\cdot\vec{v}_2 = -1-2+3=0$, $\vec{v}_1\cdot\vec{v}_3=\vec{v}_2\cdot\vec{v}_3=0$, so orthogonal
- $\vec{0}$ not in S so S is lin indep.
- 3 vectors in $\mathbb{R}^3$ & li give a basis for $\mathbb{R}^3$, so $Sp(S)=\mathbb{R}^3$
- $\|\vec{v}_1\|=\sqrt{1+4+9}=\sqrt{14}$, $\|\vec{v}_2\|=\sqrt{1+1+1}=\sqrt{3}$, $\|\vec{v}_3\|=\sqrt{25+16+1}=\sqrt{42}$

So S is not orthonormal but $S'=\left\{\frac{1}{\sqrt{14}}\vec{v}_1,\frac{1}{\sqrt{3}}\vec{v}_2,\frac{1}{\sqrt{52}}\vec{v}_3\right\}$ is.

- Pick $\vec{w}=\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$ in $\mathbb{R}^3$. Want to write its coordinates $\begin{bmatrix}a\\b\\c\end{bmatrix}$ with resp to S, ie $\vec{w}=\underline{a}\,\vec{v}_1+\underline{b}\,\vec{v}_2+\underline{c}\,\vec{v}_3$

$\rightarrow \underset{\overset{\shortparallel}{w_1+2w_2+3w_3}}{\vec{w}\cdot\vec{v}_1} = (a\vec{v}_1+b\vec{v}_2+c\vec{v}_3)\cdot\vec{v}_1 = a\underbrace{\vec{v}_1\cdot\vec{v}_1}_{\overset{\shortparallel}{\|\vec{v}_1\|^2=14}}+b\underbrace{\vec{v}_2\cdot\vec{v}_1}_{=0}+c\underbrace{\vec{v}_3\cdot\vec{v}_1}_{=0}$

so $a=\vec{w}\cdot\vec{v}_1/14=\dfrac{w_1+2w_2+3w_3}{14}$

$\longrightarrow \vec{w}\cdot\vec{v}_2 = b\|\vec{v}_2\|^2 = b\,3 \rightsquigarrow \quad b = \dfrac{\vec{w}\cdot\vec{v}_2}{3} = \dfrac{-w_1-w_2+w_3}{3}$

$\longrightarrow w\cdot\vec{v}_3 = c\|\vec{v}_3\|^2 = c\,42 \rightsquigarrow \quad c = \dfrac{\vec{w}\cdot\vec{v}_3}{42} = \dfrac{5w_1-4w_2+w_3}{42}$

Theorem: If $B=\{\vec{v}_1,\dots,\vec{v}_p\}$ is an orthogonal basis for a subspace $\mathbb{V}$ of $\mathbb{R}^n$, a $\vec{w}$ is a vector in $\mathbb{V}$, then:

$$\vec{w} = x_1\vec{v}_1 + \dots + x_p\vec{v}_p$$

where $x_1 = \dfrac{\vec{v}_1\cdot\vec{w}}{\vec{v}_1\cdot\vec{v}_1}$, $x_2 = \dfrac{\vec{v}_2\cdot\vec{w}}{\vec{v}_2\cdot\vec{v}_2}$, ..., $x_p = \dfrac{\vec{v}_p\cdot\vec{w}}{\vec{v}_p\cdot\vec{v}_p}$.

In short $[\vec{w}]_B = \begin{bmatrix} \vec{v}_1\cdot\vec{w}/\|\vec{v}_1\|^2 \\ \vec{v}_2\cdot\vec{w}/\|\vec{v}_2\|^2 \\ \vdots \\ \vec{v}_p\cdot\vec{w}/\|\vec{v}_p\|^2 \end{bmatrix}$ (vector in $\mathbb{R}^p$)

Q: How to find an orthogonal basis?

§2. Gram-Schmidt Algorithm

INPUT: A basis $B=\{\vec{w}_1,\dots,\vec{w}_p\}$ for a subspace $\mathbb{V}$ of $\mathbb{R}^n$

OUTPUT: An orthogonal basis $B'=\{\vec{u}_1,\dots,\vec{u}_p\}$ for $\mathbb{V}$.

Procedure: $\vec{u}_1 = \vec{w}_1$

$$\vec{u}_2 = \vec{w}_2 - \mathrm{proj}_{\vec{u}_1}(\vec{w}_2) = \vec{w}_2 - \underbrace{\frac{\vec{u}_1\cdot\vec{w}_2}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1}_{\text{formula for projection}}$$

$$\vec{u}_3 = \vec{w}_3 - \mathrm{proj}_{u_1}(\vec{w}_3) - \mathrm{proj}_{u_2}(\vec{w}_3)$$
$$= \vec{w}_3 - \frac{\vec{u}_1\cdot\vec{w}_3}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 - \frac{\vec{u}_2\cdot\vec{w}_3}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2$$

... continue in this way (it's an iterative procedure)

Assume we have built up to $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$. Then:

$$\vec{u}_{k+1} = \vec{w}_{k+1} - \frac{\vec{u}_1 \cdot \vec{w}_{k+1}}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 - \cdots - \frac{\vec{u}_k \cdot \vec{w}_{k+1}}{\vec{u}_k \cdot \vec{u}_k}\vec{u}_k$$

Why does it work? $\vec{u}_1 = Sp(\vec{w}_1)$

$\vec{u}_2$ in $Sp(\vec{w}_2, \vec{u}_1) = Sp(\vec{w}_2, \vec{w}_1)$

$\vec{u}_3$ in $Sp(\vec{w}_3, \vec{u}_1, \vec{u}_2) \subseteq Sp(\vec{w}_3, \vec{w}_2, \vec{w}_1)$

$\vdots$ ↳ subset of

We get $Sp(\vec{u}_1, \dots, \vec{u}_k) \subseteq Sp(\vec{w}_1, \dots, \vec{w}_k)$ for all k.

dim k ($\vec{w}$'s are li)

A direct calculation shows (1) $\vec{u}_i \cdot \vec{u}_j = 0$ for all $i \neq j$

(2) $\vec{u}_i \neq \vec{0}$ ($\vec{w}_{i+1}$ is not in $Sp(\vec{u}_1, \dots, \vec{u}_i)$)

So by the key prop on page 1: $\{\vec{u}_1, \dots, \vec{u}_k\}$ is li for all k

In particular, when $k=p$, we get p vectors $\{\vec{u}_1, \dots, \vec{u}_p\}$ that are li inside $\mathbb{W} = Sp(\vec{w}_1, \dots, \vec{w}_p)$ and $\mathbb{W}$ has dim p.

Conclusion: $\{\vec{u}_1, \dots, \vec{u}_p\}$ is a basis for $\mathbb{W}$

Example 1: $\mathbb{W} = Sp\left(\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}\right) \subset \mathbb{R}^3$

↳ this is a basis for $\mathbb{W}$

$\vec{w}_1 = \begin{bmatrix}1\\1\\2\end{bmatrix}$, $\vec{w}_2 = \begin{bmatrix}0\\2\\4\end{bmatrix}$. Use Gram-Schmidt to construct an orthogonal basis $\{\vec{u}_1, \vec{u}_2\}$

- $\vec{u}_1 = \vec{w}_1 = \begin{bmatrix}1\\1\\2\end{bmatrix}$

- $\vec{u}_2 = \vec{w}_2 - \dfrac{\vec{u}_1 \cdot \vec{w}_2}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 = \begin{bmatrix}0\\2\\4\end{bmatrix} - \dfrac{(0+2+8)}{1+1+4}\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}0\\2\\4\end{bmatrix} - \dfrac{10}{6}\begin{bmatrix}1\\1\\2\end{bmatrix}$

so $\vec{u}_2 = \begin{bmatrix} 5/3 \\ 2-5/3 \\ 4-10/3 \end{bmatrix} = \begin{bmatrix} -5/3 \\ 1/3 \\ 2/3 \end{bmatrix}$

Check $\vec{u}_1 \cdot \vec{u}_2 = -\frac{5}{3} + \frac{1}{3} + \frac{4}{3} = 0$ ✓

. To make it orthonormal, divide by $\| \ \|$: $\left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} ; \frac{1}{\sqrt{30}} \begin{bmatrix} -5 \\ 1 \\ 2 \end{bmatrix} \right\}$

Example 2: Let $\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $\vec{w}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ be 3 linearly independent vectors in $\mathbb{R}^4$.

(1) Apply Gram-Schmidt's orthogonalization procedure to get an orthogonal basis for $W = Sp(\vec{w}_1, \vec{w}_2, \vec{w}_3)$

(2) Verify that $\begin{bmatrix} 5 \\ 0 \\ -1 \\ 4 \end{bmatrix}$ is in W.

Solution: We build $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$

$\vec{u}_1 = \vec{w}_1$

$\vec{u}_2 = \vec{w}_2 - \frac{\vec{w}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix} - \frac{6}{6} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

$\vec{u}_3 = \vec{w}_3 - \frac{\vec{w}_3 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 - \frac{\vec{w}_3 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{6} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} - \frac{0}{6} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1 \\ -1/2 \\ 0 \end{bmatrix}$

Auxiliary computations needed:

$\vec{u}_1 \cdot \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = 1+1+4=6$

$\vec{u}_2 \cdot \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = 1+1+1=3$

$\vec{w}_2 \cdot \vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = 2+4=6$

$\vec{w}_3 \cdot \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = 1+2=3$

$\vec{w}_3 \cdot \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = 1-1=0$

Check the output is orthogonal: $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1/2 \\ -1 \\ -1/2 \\ 0 \end{bmatrix}$

$\vec{u}_1 \cdot \vec{u}_2 = 1 - 1 = 0$, $\vec{u}_1 \cdot \vec{u}_3 = \frac{1}{2} - \frac{1}{2} = 0$, $\vec{u}_2 \cdot \vec{u}_3 = \frac{1}{2} - 1 + \frac{1}{2} = 0$ ✓

Observe: $\vec{w}_1 = \vec{u}_1$, $\vec{w}_2 = \vec{u}_2 + \vec{u}_1$, $\vec{w}_3 = \vec{u}_3 + \frac{1}{2}\vec{u}_1$

$\vec{u}_1 = \vec{w}_1$; $\vec{u}_2 = \vec{w}_2 - \vec{w}_1$, $\vec{u}_3 = \vec{w}_3 - \frac{1}{2}\vec{w}_1$

So $Sp(\vec{u}_1) = Sp(\vec{w}_1)$, $Sp(\vec{u}_1, \vec{u}_2) = Sp(\vec{w}_1, \vec{w}_2)$ & $Sp(\vec{u}_1, \vec{u}_2, \vec{u}_3) = Sp(\vec{w}_1, \vec{w}_2, \vec{w}_3)$

(This is partly why the algorithm works!)

Now: $\vec{b} = \begin{bmatrix} 5 \\ 0 \\ -1 \\ 4 \end{bmatrix}$ in $\mathbb{R}^4$. If $\vec{b}$ in $Sp(\vec{w}_1, \vec{w}_2, \vec{w}_3) = Sp(\vec{u}_1, \vec{u}_2, \vec{u}_3)$

we must find real numbers x_1, x_2, x_3 with $\vec{b} = \underline{\underline{x_1}}\,\vec{u}_1 + \underline{\underline{x_2}}\,\vec{u}_2 + \underline{\underline{x_3}}\,\vec{u}_3$

Advantage: $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is orthogonal! We compute x_1, x_2, x_3 by taking dot product with $\vec{u}_1$, $\vec{u}_2$ & $\vec{u}_3$, respectively.

$$\rightarrow \vec{b} \cdot \vec{u}_1 = x_1 \underbrace{\vec{u}_1 \cdot \vec{u}_1}_{=6} + x_2 \underbrace{\vec{u}_2 \cdot \vec{u}_1}_{=0} + x_3 \underbrace{\vec{u}_3 \cdot \vec{u}_1}_{=0}$$

$$\text{So } x_1 = \frac{\vec{b} \cdot \vec{u}_1}{6} = \frac{5 - 1 + 8}{6} = \frac{12}{6} = 2$$

$$\rightarrow b \cdot \vec{u}_2 = x_1 \underbrace{\vec{u}_2 \cdot \vec{u}_1}_{=0} + x_2 \underbrace{\vec{u}_2 \cdot \vec{u}_2}_{=3} + x_3 \underbrace{\vec{u}_3 \cdot \vec{u}_2}_{=0}$$

$$\text{So } x_2 = \frac{\vec{b} \cdot \vec{u}_2}{3} = \frac{5 + 1}{3} = \frac{6}{3} = 2$$

$$\rightarrow b \cdot \vec{u}_3 = x_1 \underbrace{\vec{u}_1 \cdot \vec{u}_3}_{=0} + x_2 \underbrace{\vec{u}_2 \cdot \vec{u}_3}_{=0} + x_3 \underbrace{\vec{u}_3 \cdot \vec{u}_3}_{= \frac{1}{4} + 1 + \frac{1}{4} = \frac{3}{2}}$$

$$\text{So } x_3 = \frac{b\, \vec{u}_3}{3/2} = \frac{2}{3}\left(\frac{5}{2} + \frac{1}{2}\right) = 2$$

Check: $\begin{bmatrix} 5 \\ 0 \\ -1 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1/2 \\ -1 \\ -1/2 \\ 0 \end{bmatrix}$

$5 = 2 + 2 + 1$ ✓
$0 = 2 - 2$ ✓
$-1 = 2 - 2 - 1$ ✓
$4 = 4 + 0 + 0$ ✓