Lecture XXI : §3.7 Linear Transformations

- **So far:** we've studied $\mathbb{R}^n$ & subspaces of $\mathbb{R}^n$ (l.i., spanning sets, bases, dim, coordinates w.r.t bases)
  - matrices $\rightsquigarrow$ $\mathcal{N}(A)$, $\mathcal{R}(A)$, RowSp(A).

- **Our next goal:** relate subspaces (of $\mathbb{R}^n$) via linear functions called linear transformations.
- Recall (from Calculus) that a function $\boxed{F: \mathbb{R}^n \longrightarrow \mathbb{R}^m}$ takes inputs from $\mathbb{R}^n$ & gives outputs in $\mathbb{R}^m$.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^n \rightsquigarrow \boxed{F} \rightsquigarrow \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ f_2(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix} \text{ in } \mathbb{R}^m$$

For example: (1) $n = m = 1$ (Calculus I) functions of one variable $f(x)$

$$x \text{ in } \mathbb{R} \rightsquigarrow f(x) \text{ in } \mathbb{R}$$

Eg: $\sin(x)$, x^2+4, e^x, $4x$

(2) $n=2$, $m=1$ (Calculus III) functions of 2 variables $f(x,y)$

$$\begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 \rightsquigarrow f(x,y) \text{ in } \mathbb{R}$$

Eg: $f(x,y) = 2x+4y$, x^3-y^2 , $\sin(x+y)$, ...

(3) $n=3$, $m=3$ called Vector Fields in Calculus III

Three ($=m$) functions $f(x,y,z)$, $g(x,y,z)$, $h(x,y,z)$ in $3\,(=n)$ variables

$$\text{Eg} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ in } \mathbb{R}^3 \rightsquigarrow \begin{bmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{bmatrix} = \begin{bmatrix} 2xy+4 \\ \cos(x^3 z) \\ 9y^2-4z \end{bmatrix} \text{ in } \mathbb{R}^3$$

§1. Linear Transformations:

Linear transformations are functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ (or from subspaces of $\mathbb{R}^n$ to $\mathbb{R}^m$) which respect the vector space structure (ie, addition & scalar multiplication)

Definition: A linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfying:

1. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for $\vec{u}, \vec{v}$ in $\mathbb{R}^n$
2. $T(c\vec{u}) = cT(\vec{u})$ for $\vec{u}$ in $\mathbb{R}^n$, c scalar.

(Can restrict to subspaces $T: W \longrightarrow \mathbb{R}^m$ because of (S2) & (S3) taking $\vec{u}, \vec{v}$ in W)

Almost none of the above examples are linear transformations. Only $f(x,y) = 2x+4y : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is. Write $T: \mathbb{R}^2 \longrightarrow \mathbb{R}$ $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2x+4y$

Check it's linear.

(1) $T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = 2(x_1+x_2) + 4(y_1+y_2)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \checkmark$

$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = 2x_1 + 4y_1 + 2x_2 + 4y_2 = 2(x_1+x_2) + 4(y_1+y_2)$

(2) $T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = 2(\underline{\underline{c}}x) + 4(\underline{\underline{c}}y) = \underline{\underline{c}}(2x+4y) = cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ ✓

The point is: in the picture $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \rightsquigarrow \boxed{F} \rightsquigarrow \begin{bmatrix} f_1(x_1, \ldots x_n) \\ \vdots \\ f_m(x_1, \ldots x_n) \end{bmatrix}$

EVERY $f_1, f_2, \ldots, f_m$ has to be a linear expression in $x_1, \ldots x_n$ without a constant term if F is a lin. transf.

Example (1) Let $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ 2y+1 \end{bmatrix}$

This is NOT a linear Transformation

$$F\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = F\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1+x_2)-(y_1+y_2) \\ 2(y_1+y_2)+1 \end{bmatrix}$$

NOT equal

$$F\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + F\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1-y_1 \\ 2y_1+1 \end{bmatrix} + \begin{bmatrix} x_2-y_2 \\ 2y_2+1 \end{bmatrix} = \begin{bmatrix} x_1+x_2-y_1-y_2 \\ 2(y_1+y_2)+2 \end{bmatrix}$$

(2) $G: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ 2y \end{bmatrix}$ is a linear transf.

- $$G\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = G\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2-(y_1+y_2) \\ 2(y_1+y_2) \end{bmatrix}$$

  $$G\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + G\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1-y_1 \\ 2y_1 \end{bmatrix} + \begin{bmatrix} x_2-y_2 \\ 2y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2-(y_1+y_2) \\ 2(y_1+y_2) \end{bmatrix}$$ ✓

- $$G\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = G\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} cx-cy \\ 2(cy) \end{bmatrix} = c\begin{bmatrix} x-y \\ 2y \end{bmatrix} = c\,G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$ ✓

- Find a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ so that $G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Soln: $\quad x-y=1$, $\quad 2y=1 \rightsquigarrow y = \frac{1}{2} \rightsquigarrow x = 1+y = 1+\frac{1}{2} = \frac{3}{2}$ $\quad$ so $G\left(\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Key: If $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transf then $\underset{\text{in } \mathbb{R}^n}{T(\vec{0})} = \underset{\text{in } \mathbb{R}^m}{\vec{0}}$

Why? $\quad T(\underset{\text{in } \mathbb{R}^n}{\vec{0}}) = T(0\cdot\vec{0}) \underset{\text{linear}}{=} 0\cdot \underset{\text{in } \mathbb{R}^m}{T(\vec{0})} = \underset{\text{in } \mathbb{R}^m}{\vec{0}}$

Example above: $\quad F\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0-0 \\ 2\cdot 0+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ confirms F is not a linear transf.

(take $x=y=0$ in the formula)

§2 Main examples:

Warm-up: Every linear transf $T: \mathbb{R} \longrightarrow \mathbb{R}$ is determined by a number

a in $\mathbb{R}$: $\quad T(x) = ax. \quad (a = T(1))$

- Check this is a linear transf :

① $T(x_1 + x_2) = a(x_1+x_2) = ax_1 + ax_2 = T(x_1) + T(x_2)$ ✓

② $T(cx) = a(cx) = c(ax) = cT(x)$ ✓

- Now, check any linear transformation T is completely determined by $T(1)$

$$T(x) = \underset{\text{think of } x \text{ as a scalar}}{T(x \cdot 1)} = x\,\boxed{T(1)}^{=a}$$

, so T is of the form x in $\mathbb{R} \rightsquigarrow ax$ in $\mathbb{R}$

- <u>Main example:</u> Fix a matrix A of size $m \times n$

Define $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$

(Remember the picture $\quad \underset{\vec{x}}{\mathbb{R}^n} \rightsquigarrow \boxed{\begin{matrix} A \\ m\times n \text{ matrix}\end{matrix}} \rightsquigarrow \underset{A\vec{x}}{\mathbb{R}^m}$)

Then, T is a linear transformation by the algebraic properties of matrix operations (§1.6). Indeed

① $T(\vec{u} + \vec{v}) = A(\vec{u} + \vec{v}) \underset{\text{Distrib}}{=} A\vec{u} + A\vec{v} = T(\vec{u}) + T(\vec{v})$

② $T(c\vec{u}) = A(c\vec{u}) \underset{\text{scalars jump}}{=} c(A\vec{u}) = cT(\vec{u})$.

<u>Example</u> (1) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}$ 2×3 $\rightsquigarrow$ $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_3 \\ -x_2 + 3x_3 \end{bmatrix}$$

<u>Next Time</u>: Linear transf $T: \mathbb{R}^n \to \mathbb{R}^m$ are completely determined by their values on a basis for $\mathbb{R}^n$. (Same is true for $T: \mathbb{W} \longrightarrow \mathbb{R}^m$ for $\mathbb{W}$ a subspace of $\mathbb{R}^n$)