Lecture XXII: §3.7 Linear Transformations

Definition: A linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfying:

① $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for $\vec{u}, \vec{v}$ in $\mathbb{R}^n$

② $T(c\vec{u}) = cT(\vec{u})$ for $\vec{u}$ in $\mathbb{R}^n$, c scalar.

§1 Examples & Properties:

Main example: A $m\times n$ matrix $\rightsquigarrow$ $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $T(\vec{x}) = A\vec{x}$

Example: (1) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}$ 2×3 $\rightsquigarrow$ $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_3 \\ -x_2 + 3x_3 \end{bmatrix}$$

Note
$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \text{Col}_1 A$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \text{Col}_2 A$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \text{Col}_3 A$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T\left(x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \underset{①\&②}{=} x_1 T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + x_3 T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= x_1 \text{Col}_1 A + x_2 \text{Col}_2 A + x_3 \text{Col}_3 A$$

$$= A\vec{x}$$

$\rightsquigarrow$ So T is determine once we know $T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)$.

(2) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x - 3y \\ 2x \\ 7y \end{bmatrix}$

Then $T(\vec{x}) = A\vec{x}$ for the following 3×2 matrix $A = \begin{bmatrix} 5 & -3 \\ 2 & 0 \\ 0 & 7 \end{bmatrix}$

$\left(\text{Col}_1 A = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \quad \text{Col}_2 A = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right)$

Prop: Fix W of subspace of $\mathbb{R}^n$ of dim p. Fix a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_p\}$ for W.

Then any linear transformation $T: W \longrightarrow \mathbb{R}^m$ is completely determined by where it sends $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$.

Why? Write

$$\left.\begin{aligned} \vec{b}_1 &= T(\vec{v}_1) \\ \vec{b}_2 &= T(\vec{v}_2) \\ &\vdots \\ \vec{b}_p &= T(\vec{v}_p) \end{aligned}\right\} \text{ p vectors in } \mathbb{R}^m$$

Any $\vec{v}$ in V can be written in a <u>unique</u> way as a linear comb of the basis $\mathcal{B}$:

$$\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_p \vec{v}_p$$

($a_1, a_2, \dots, a_p$ are numbers $= [\vec{v}]_{\mathcal{B}}$)

Then, using the 2 defining properties of T we get

$$\begin{aligned} T(\vec{v}) &= T(a_1 \vec{v}_1 + (a_2 \vec{v}_2 + \dots + a_p \vec{v}_p)) \\ &= a_1 T(\vec{v}_1) + T(a_2 \vec{v}_2 + \dots + a_p \vec{v}_p) \\ &= \dots = a_1 T(\vec{v}_1) + a_2 T(\vec{v}_2) + \dots + a_p T(\vec{v}_p) \\ &= a_1 \vec{b}_1 + a_2 \vec{b}_2 + \dots + a_p \vec{b}_p \end{aligned}$$

Example $W = \mathrm{Sp}\left(\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$ in $\mathbb{R}^3$

Note $\{\vec{v}_1, \vec{v}_2\}$ are li, so they give a basis for W.

A linear transf $T: W \longrightarrow \mathbb{R}^2$ <u>can</u> be defined by $T(\vec{v}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $T(\vec{v}_2) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$T\left(\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}\right) = ?$ To answer, we write $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} = 5\vec{v}_1 + 4\vec{v}_2$.

So $T\left(\begin{bmatrix}5\\-1\\4\end{bmatrix}\right) = 5T(\vec{v}_1) + 4T(\vec{v}_2) = 5\begin{bmatrix}1\\2\end{bmatrix} + 4\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}13\\26\end{bmatrix}$

SPECIAL CASE: $V = \mathbb{R}^n$

- In particular, if $V = \mathbb{R}^n$ & $B = \{\vec{e}_1, \ldots, \vec{e}_n\}$, then $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear is completely determined by $T(\vec{e}_1), \ldots, T(\vec{e}_n)$

If we build an $m \times n$ matrix $A = [\vec{b}_1 \; \vec{b}_2 \; \ldots \; \vec{b}_n]$ with $\vec{b}_i = T(\vec{e}_i)$

Then $T(\vec{x}) = A\vec{x}$

Same idea: $T\left(\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix}\right) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n)$

$$= x_1 \underbrace{T(\vec{e}_1)}_{=\vec{b}_1} + x_2 \underbrace{T(\vec{e}_2)}_{=\vec{b}_2} + \cdots + x_n \underbrace{T(\vec{e}_n)}_{=\vec{b}_n} = A\vec{x}.$$

Conclusion: Linear Transf $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\xleftrightarrow{\text{1-to-1}}$ $m \times n$ matrices

Important Obs: If the input vectors are l.d, we have to check their images under T satisfy the same relation.

Example: Find a linear transf with $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}5\\4\\6\end{bmatrix}$

$T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}6\\0\\-1\end{bmatrix}$.

Solution: First 2 vectors are a basis B for $\mathbb{R}^2$ so they determine T uniquely:

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = T\left((x-y)\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}1\\1\end{bmatrix}\right) \qquad \left[\begin{bmatrix}x\\y\end{bmatrix}\right]_B = \begin{bmatrix}x-y\\y\end{bmatrix}$$

$$= (x-y)\,T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + y\,T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = (x-y)\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}5\\4\\6\end{bmatrix}$$

$$= \begin{bmatrix}x+4y\\2x+2y\\x+3y\end{bmatrix} = \begin{bmatrix}1&4\\2&2\\1&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Check $T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}-1+4\\-2+2\\-1+3\end{bmatrix} = \begin{bmatrix}3\\0\\2\end{bmatrix} \neq \begin{bmatrix}6\\0\\-1\end{bmatrix}$ so no T can be constructed!

Issue: $-2\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

BUT $-2\begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}5\\4\\6\end{bmatrix}-\begin{bmatrix}6\\0\\-1\end{bmatrix}\neq\begin{bmatrix}0\\0\\0\end{bmatrix}$

Applying T should preserve the relation, but it doesn't

§2. Matrix Representation

Just like for $\mathbb{R}^n$ & $\mathbb{R}^m$, a linear transformation $T: V \to W$ where V & W are subspaces of $\mathbb{R}^n$ & $\mathbb{R}^m$, respectively can ALWAYS be represented by a matrix.

VERY IMPORTANT: The representation of $T: V \to W$ depends on a choice of bases: one for V & one for W.

For $V = \mathbb{R}^n$ & $W = \mathbb{R}^m$ our "natural choices" is $\{\vec{e}_1, \ldots, \vec{e}_n\}$ for $\mathbb{R}^n$ & $\{\vec{e}_1, \ldots, \vec{e}_m\}$ for $\mathbb{R}^m$

How does it work?

Fix $B_1 = \{\vec{v}_1, \ldots, \vec{v}_p\}$ a basis for V $(\dim V = p)$

$B_2 = \{\vec{w}_1, \ldots, \vec{w}_q\}$ ——— W $(\dim W = q)$

To build the matrix, we need to write the coordinates of $T(\vec{v}_1), \ldots, T(\vec{v}_p)$ wrt the basis B_2. More precisely:

$$T(\vec{v}_1) = a_{11}\vec{w}_1 + \cdots + a_{q1}\vec{w}_q$$
$$T(\vec{v}_2) = a_{12}\vec{w}_1 + \cdots + a_{q2}\vec{w}_q$$
$$\vdots$$
$$T(\vec{v}_p) = a_{1p}\vec{w}_1 + \cdots + a_{qp}\vec{w}_q$$

$$\rightsquigarrow A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & & & \\ a_{q1} & a_{q2} & \cdots & a_{qp}\end{bmatrix}$$

$q \times p$ matrix

How does A encode the linear transformation we started from?

1. Given $\vec{v}$ in V, we express $\vec{v} = x_1\vec{v}_1 + \cdots + x_p\vec{v}_p$ $\left([\vec{v}]_B = \begin{bmatrix}x_1\\ \vdots \\ x_p\end{bmatrix}\right)$

2. Write $\begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$

3. $T(\vec{v}) = y_1 \vec{w}_1 + \cdots + y_q \vec{w}_q$ $\quad \left([T(\vec{v})]_{B_2} = \begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} \right)$

In symbols: $[T(\vec{v})]_{B_2} = A\, [\vec{v}]_B$ $\quad$ (Write $A = [T]_{BB_2}$)

<u>Examples:</u> $\mathbb{V} = Sp\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$ subspace of $\mathbb{R}^3$ of dim $= 2$, $\mathbb{W} = \mathbb{R}^2$

$T : \mathbb{V} \longrightarrow \mathbb{W}$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x+y \\ y+z \end{bmatrix}$

- Basis $B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{V}$; Basis $B_2 = \left\{ \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{W}$

$$T(\vec{v}_1) = \begin{bmatrix} 1+0 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{e}_1 + \vec{e}_2$$
$$T(\vec{v}_2) = \begin{bmatrix} 0+1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{e}_1 + 2\vec{e}_2$$
$$\rightsquigarrow [T]_{B_1 B_2} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

- Change the basis B_1 to $B'_1 = \left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} ; \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$, where $\vec{u}_1 = \vec{v}_1 - \vec{v}_2$ and $\vec{u}_2 = \vec{v}_1 + \vec{v}_2$

$$T(\vec{u}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\vec{e}_2$$
$$T(\vec{u}_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\vec{e}_1 + 3\vec{e}_2$$
$$\rightsquigarrow [T]_{B'_1 B_2} = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$$

<u>Check:</u> $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ in $\mathbb{V}$. So from the formula for T we get $T(\vec{v}) = \begin{bmatrix} 2+2 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$\vec{v} = 2\vec{v}_1 + 2\vec{v}_2$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_{[T]_{B_1 B_2}} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 6 \end{bmatrix}}_{[T(\vec{v})]_{B_2}}$$

$\vec{v} = 2\vec{u}_2 = 0 \cdot \vec{u}_1 + 2 \cdot \vec{u}_2$

$$\underbrace{\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}}_{[T]_{B'_1 B_2}} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 6 \end{bmatrix}}_{[T(\vec{v})]_{B_2}}$$