Lecture XXIII: §3.7 Rank & Nullity of Linear Transformations
§5.1–5.2 Abstract Vector spaces

Recall:

Definition: A linear transformation $T: \mathbb{V} \longrightarrow \mathbb{W}$ is a function from $\mathbb{V}$ (subspace of $\mathbb{R}^n$) to $\mathbb{W}$ (subspace of $\mathbb{R}^m$) satisfying:

① $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for $\vec{u}, \vec{v}$ in $\mathbb{R}^n$

② $T(c\vec{u}) = cT(\vec{u})$ for $\vec{u}$ in $\mathbb{R}^n$, c scalar.

If $B_1 = \{\vec{v}_1, \dots, \vec{v}_p\}$ is a basis for $\mathbb{V}$ then
$B_2 = \{\vec{w}_1, \dots, \vec{w}_q\}$ ______ $\mathbb{W}$

- T is completely determined by a choice of vectors $T(\vec{v}_1), \dots, T(\vec{v}_p)$ in $\mathbb{W}$ (& we have freedom of choice)
- T is represented by a $q \times p$ matrix $A = [T]_{B_1 B_2}$ (depending on the choice of bases B_1 & B_2). Indeed:

$$[T(\vec{v})]_{B_2} = A\,[\vec{v}]_{B_1}$$

Ex: $\mathbb{V} = \mathbb{R}^n$, $\mathbb{W} = \mathbb{R}^m$ $B_1 = \{\vec{e}_1, \dots, \vec{e}_n\}$, $B_2 = \{\vec{e}_1, \dots, \vec{e}_m\}$ $T(\vec{x}) = A\vec{x}$ then $A = [T]$.
(& conversely, T depends on A)

§1. Rank-Nullity

The Nullspace & Range of a linear transformation are defined analogous to their definition for matrices:

Let $T: \mathbb{V} \longrightarrow \mathbb{R}^m$ be a linear transformation

Definition: Null Space of T: $\mathcal{N}(T) = \{\vec{v}$ in $\mathbb{V}$ such that $T(\vec{v}) = \vec{0}\}$

Range of T: $\mathcal{R}(T) = \{\vec{w}$ in $\mathbb{R}^m$ ______ $\vec{w} = T(\vec{v})$ for some $\vec{v}$ in $\mathbb{V}\}$

Both are <u>subspaces</u> $\rightsquigarrow$ we can replace $T: \mathbb{V} \to \mathbb{R}^m$

by $T: \mathbb{V} \to \mathbb{W}$ with $\mathbb{W} = \mathcal{R}(T)$.

<u>Definition:</u> nullity $(T) = \dim(\mathcal{N}(T))$ rank $(T) = \dim(\mathcal{R}(T))$

Later in this course we will reprove the <u>rank-nullity Theorem</u>:

$$\boxed{\text{nullity } (T) + \text{rank } (T) = \dim \mathbb{V}}$$

Special case: $\mathbb{V} = \mathbb{R}^n$ & $T: \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\vec{x}) = A\vec{x}$

Then $\mathcal{N}(T) = \mathcal{N}(A)$, $\mathcal{R}(T) = \mathcal{R}(A)$

$\Downarrow$ dim $\Downarrow$ dim

nullity $(T) =$ nullity (A) rank $(T) =$ rank (A)

Rank-Nullity for T <u>is</u> the rank-nullity theorem for A.

§2 Abstract Vector Spaces:

So far, we have consider $\mathbb{R}^n$ and its subspaces:

- Two main operations on $\mathbb{R}^n$: <u>addition</u> and <u>scalar multiplication</u> satisfying 8 properties (Assoc., Distrib, Neutral Element $\vec{0}$, additive inverse) [§3.2]
- A subspace $V \subseteq \mathbb{R}^n$ was defined as subsets containing $\vec{0}$ and "closed under addition & scalar multiplication".

<u>Punchline</u>: There are many mathematical objects which admit these 2 operations with the same 10 properties. We call them vector spaces

<u>EXAMPLES</u>: $\mathbb{F} = \{ f: \mathbb{R} \to \mathbb{R} \}$ (set of all functions of one variable)

(eg $\sin(x)$, $\cos(x)$, e^x, $x^4 + x^2 + 1$, ... are all "elements" of $\mathbb{F}$)

- functions can be added together to get new functions
- ________ scaled by a real number.

Eg: $\sin(x) + e^x + \cos(4x)$

$\sqrt{2}\sin x$, $4e^x$, $\frac{1}{9}\cos(2x)$ are all elements of $\mathbb{F}$.

Adding additional properties to our functions will produce subspaces

$$\mathbb{F}^0 = \{ f : \mathbb{R} \longrightarrow \mathbb{R} \text{ continuous} \} \subsetneq \mathbb{F}$$

$$\mathbb{F}^1 = \{ f : \mathbb{R} \longrightarrow \mathbb{R} \text{ differentiable} \} \subsetneq \mathbb{F}^0$$

Formal definition: A vector space (over $\mathbb{R}$) is a set V with 2 operations

- addition $V \times V \longrightarrow V$ (elements of V are called "vectors")

  $(\vec{v}_1, \vec{v}_2) \longmapsto \vec{v}_1 + \vec{v}_2$

- scalar multiplication: $\mathbb{R} \times V \longrightarrow V$

  $(a, \vec{v}) \longmapsto a \cdot \vec{v}$

These 2 operations are required to satisfy the following 8 properties

Addition:

(A1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ for all $\vec{u}, \vec{v}$ in V

(A2) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ for all $\vec{u}, \vec{v}, \vec{w}$ in V

(A3) There is an element $\vec{0}$ in V (Neutral Element) satisfying $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$ for all $\vec{v}$ in V

(A4) Given $\vec{v}$ in V we can find "$-\vec{v}$" (additive inverse) satisfying $\vec{v} + "-\vec{v}" = \vec{0}$

Scalar Multiplication:

(M1) $a(b\vec{v}) = (ab)\vec{v}$ for every a, b in $\mathbb{R}$, $\vec{v}$ in V

(M2) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ for every a in $\mathbb{R}$, $\vec{v}$ in V

(M3) $(a+b)\vec{v} = a\vec{u} + b\vec{v}$ —— a, b in $\mathbb{R}$, $\vec{v}$ in V

(M4) $1 \cdot \vec{v} = \vec{v}$ for every $\vec{v}$ in V

Old Examples: • $\mathbb{R}^N$ is a vector space with usual + & scalar mult. [§3.2]

• $Mat_{m\times n}(\mathbb{R})$ = set of all $m\times n$ matrices is a vector space by §1.6

• Null Space, Range & Row Space of a matrix are vector spaces.

New Examples: • $\mathcal{F}$ = all cont functions $\mathbb{R}\to\mathbb{R}$ ($\vec{0}$ = zero function)

• $C([0,1])$ = all continuous functions defined on the interval $[0,1]$ ($0\leq x\leq 1$)

• $\mathcal{P}_n$ = all polynomials of degree at most n

$= \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \; : \; a_0, \ldots, a_n \text{ in } \mathbb{R} \text{ arbitrary}\}$

Addition

$$\begin{array}{rl} & a_0 + a_1 x + \cdots + a_n x^n \\ + & b_0 + b_1 x + \cdots + b_n x^n \\ \hline & (a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n \end{array}$$

("Term-by-Term")

Scalar Multiplication: $c(a_0 + a_1 x + \cdots + a_n x^n) = (ca_0) + (ca_1)x + \cdots + (ca_n)x^n$

Neutral Element: 0 polynomial $= 0 + 0x + \cdots + 0\cdot x^n$

§3. Subspaces

Fix a vector space V

Definition: A subset W of V is called a subspace if

(S1) $\vec{0}$ in W (here $\vec{0}$ is Neutral Element in V)

(S2) $\vec{v_1}, \vec{v_2}$ in W implies $\vec{v_1} + \vec{v_2}$ in W

(S3) $\vec{v}$ in W, a in $\mathbb{R}$ implies $a\cdot\vec{v}$ in W

Old Examples: Given a matrix A of size $m\times n$

• Nullspace (A) is a subspace of $\mathbb{R}^n$

• Range of $A = \mathcal{R}(A)$ is a subspace of $\mathbb{R}^m$

New Examples: $V = \mathbb{F}$ = functions of one variable

- $\mathcal{P}_n$ = polynomials of degree at most n is a subspace of $\mathbb{F}$
- $S = \{ f(x) : f'' = -f \}$ is a subspace of $\mathbb{F}$.

- Non-examples:

① $S = \{ f(x) \text{ in } \mathbb{F} : \int_0^1 f(x)\,dx = 1 \}$

The zero function is not in S, so (S1) fails

② $S = \{$ polynomials P in $\mathcal{P}_n$ with $P(0) = 1 \}$ is not a subspace because (S1) fails (zero polynomial is not in S.

③ $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc \neq 0 \right\} \subset Mat_{2\times 2}(\mathbb{R})$ is not a subspace

because (S1) fails $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is NOT in S.

$X = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 0 \right\} \subseteq Mat_{2\times 2}(\mathbb{R})$ is not a subspace

because (S2) fails : $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ both in X but their sum is $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and I_2 is NOT in X.