Lecture XXIV: §5.1–5.2 Abstract Vector spaces
§5.3 Subspaces

Recall. A set V with $+$ & scalar multiplication is a vector space if it satisfies 8 properties (including neutral element $\vec{0}$, etc)

Elements of V are called vectors & elements of $\mathbb{R}$ are called scalars

Examples: $\mathrm{Mat}_{m\times n} = \{m\times n \text{ matrices}\}$ & $\mathcal{P}_n = \{\text{polynomials of deg} \leq n\}$
$= \{a_0 + a_1 x + \cdots + a_n x^n : a_i \text{ free}\}$

. A subset W of V is a subspace if $\vec{0}$ of V is in W & W is closed under $+$ & scalar mult.

§1. Useful Properties of Vector Spaces:

Fix V a vector space

① Cancellation: If $\vec{u} + \vec{v} = \vec{u} + \vec{w}$ then $\vec{v} = \vec{w}$

(Reason: Add $(-\vec{u})$ to both sides of the equation & use $\vec{0} + \vec{v} = \vec{v}$, $\vec{0} + \vec{w} = \vec{w}$)

② Neutral Element is unique: Meaning if there are two elements $\vec{0}$ & $\vec{0}'$ satisfying $\vec{0} + \vec{v} = \vec{v}$, $\vec{0}' + \vec{v} = \vec{v}$ for all $\vec{v}$, then $\vec{0} = \vec{0}'$

(Reason $\vec{0} \underset{\vec{0}' \text{ neutral}}{=} \vec{0} + \vec{0}' \underset{\vec{0} \text{ neutral}}{=} \vec{0}'$)

③ Additive Inverse is unique: Meaning: given $\vec{v}$ in V, if there are 2 elements $\vec{u}$ & $\vec{w}$ with $\vec{u} + \vec{v} = \vec{0}$, $\vec{w} + \vec{v} = \vec{0}$, then $\vec{u} = \vec{w}$

(Reason $\vec{u} = \vec{u} + \vec{0} = \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} = \vec{0} + \vec{w} = \vec{v}$)

④ $0 \cdot \vec{v} = \vec{0}$ for any $\vec{v}$ in V

(Reason: $0\cdot\vec{v} = (0+0)\vec{v} = 0\cdot\vec{v} + 0\cdot\vec{v}$ so $\vec{0} + 0\cdot\vec{v} = 0\cdot\vec{v} + 0\cdot\vec{v}$
so by cancellation we get $\vec{0} = 0\cdot\vec{v}$)

⑤ $a\cdot\vec{\mathbb{0}} = \vec{\mathbb{0}}$ for every a in $\mathbb{R}$

(Reason: $a\cdot\vec{\mathbb{0}} = a\cdot(\vec{\mathbb{0}}+\vec{\mathbb{0}}) = a\cdot\vec{\mathbb{0}} + a\cdot\vec{\mathbb{0}}$, so $\vec{\mathbb{0}} + a\cdot\vec{\mathbb{0}} = a\cdot\vec{\mathbb{0}} + a\cdot\vec{\mathbb{0}}$
so by cancellation $\vec{\mathbb{0}} = a\cdot\vec{\mathbb{0}}$

⑥ "$-\vec{v}$" $= (-1)\cdot\vec{v}$ for every $\vec{v}$ in V

(Reason: $\vec{v} + (-1)\cdot\vec{v} = 1\cdot\vec{v} + (-1)\cdot\vec{v} = (1-1)\cdot\vec{v} = 0\cdot\vec{v} = \vec{\mathbb{0}}$
by ④ so $(-1)\cdot\vec{v}$ satisfies the defining property of Additive Inverse. By the uniqueness, we get $(-1)\cdot\vec{v} =$ "$-\vec{v}$".)

§2. Examples of Subspaces:

- Main examples of vector spaces:
  - $\mathrm{Mat}_{m\times n}$ = set of all $m\times n$ matrices
  - $\mathbb{F}$ = set of all functions of 1 variable
  - $\mathcal{P}_n$ = set of all polynomials of degree at most n
  - $C[0,1]$ = set of all continuous functions defined on $[0,1] = \{0\leq x\leq 1\}$

Note: • $\mathcal{P}_n$ is a subspace of $\mathbb{F}$. Also $\mathcal{P}_n$ is a subspace of $C[0,1]$
• $C[0,1]$ is not a subspace of $\mathbb{F}$.

Example Fix $m = n \geq 2$ & $\mathrm{Mat}_{n\times n}$ = all square $n\times n$ matrices

$S = \{A \text{ in } \mathrm{Mat}_{n\times n} : A^T = A\}$ symmetric $(n\times n)$ matrices

Claim: S is a subspace of $\mathrm{Mat}_{n\times n}$

(S1) $\mathbb{0} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$ is in S

(S2) A, B in S, then $(A+B)^T = A^T + B^T = A + B$ so in S

(S3) A in S c in $\mathbb{R}$, then $(cA)^T = cA^T = cA$ so in S

Intuitive idea: "subspaces are defined by linear homogeneous eqns"

Examples ① $X = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ab = 0 \}$ is not a subspace of $\mathrm{Mat}_{2\times 2}$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ both in X but $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is not.

② $Y = \{ p(x)$ in $\mathcal{P}_3$ with $p''(0) = 5 \}$ is not a subspace of $\mathcal{P}_3$

not homogeneous

Zero polynomial not in Y.

Fix: $W = \{ p(x)$ in $\mathcal{P}_3$ with $p''(0) = 0 \}$ is a subspace of $\mathcal{P}_3$

Reason:
- $\mathbb{0}''(x) = 0$
- $(f(x) + g(x))'' = f''(x) + g''(x)$
- $(c\,f(x))'' = c\,f''(x)$

§3. Spanning Sets: V = fixed vector space

Def: $S = \{ \vec{v}_1, \dots, \vec{v}_p \}$ a set of "vectors" in V

$Sp(S)$ = span of S = set of all linear comb of $\vec{v}_1, \dots, \vec{v}_p$

$= \{ a_1\vec{v}_1 + \dots + a_p\vec{v}_p$ where $a_1, \dots, a_p$ is arbitrary$\}$

Prop: $Sp(S)$ is a subspace of V

Proof: Same reasons as for spans in $\mathbb{R}^n$.

(S1) $\vec{\mathbb{0}} = \underbrace{\vec{\mathbb{0}} + \dots + \vec{\mathbb{0}}}_{n \text{ times}} = 0\cdot\vec{v}_1 + \dots + 0\cdot\vec{v}_p$ in $Sp(S)$

(S2)
$$\begin{aligned} \vec{u} &= a_1\vec{v}_1 + \dots + a_p\vec{v}_p \quad \text{in } Sp(S) \\ + \ \vec{w} &= b_1\vec{v}_1 + \dots + b_p\vec{v}_p \\ \hline \vec{u} + \vec{w} &= (a_1+b_1)\vec{v}_1 + \dots + (a_p+b_p)\vec{v}_p \quad \text{in } Sp(S) \end{aligned}$$

Here we use (M3), (A1)

(S3) $\vec{u} = a_1\vec{v}_1 + \cdots + a_p\vec{v}_p$ in $Sp(S)$ $\rightsquigarrow$ $c\vec{u} = (ca_1)\vec{v}_1 + \cdots + (ca_p)\vec{v}_p$ in $Sp(S)$

<u>Definition</u>: Fix a subspace W of V & $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ a finite set of vectors in W. If $Sp(S) = W$, we say S is a <u>spanning set</u> for W

<u>Examples</u>: ① $\mathcal{P}_3 = V = W = \{a_0 1 + a_1 x + a_2 x^2\}$
$= Sp(1, x, x^2)$

② $W = \{p \text{ in } \mathcal{P}_3 : p''(0) = 0\}$ subspace of $\mathcal{P}_3 = V$

$p(x) = a_0 + a_1 x + a_2 x^2$
$p'(x) = a_1 + 2a_2 x$ so $p''(0) = 0$ becomes $a_2 = 0$
$p''(x) = 2a_2$

So $W = \mathcal{P}_2 = Sp(1, x)$

③ $W_2 = \{p \text{ in } \mathcal{P}_3 : p'(0) = 0\}$ is a subspace of $\mathcal{P}_3 = V$

Condition becomes $a_1 + 2a_2 \cdot 0 = 0$, so $a_1 = 0$

$W_2 = \{a_0 + a_2 x^2 : a_0, a_2 \text{ free}\} = Sp(1, x^2)$

<u>§4. Linear Independence</u>: Fix V a vector space

<u>Def</u>: Given a set $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ of vectors in V we say

- S is <u>linearly dependent</u> if we can find scalars $a_1, a_2, \ldots, a_p$ not all zero satisfying:

$$(*) \quad a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_p\vec{v}_p = \vec{0} \quad (\text{in } V)$$

(We view this equation as a "dependence relation among $\vec{v}_1, \ldots, \vec{v}_p$)

- S is <u>linearly independent</u> if (*) only has one solution: $a_1 = a_2 = \cdots = a_p = 0$

§5. Bases

<u>Def</u>: A set of vectors $B = \{\vec{v}_1, \ldots, \vec{v}_p\}$ of a vector space V is a <u>basis</u> for V if
- B is a spanning set for V ($Sp(B) = V$)

&
- B is linearly indep

<u>Examples</u>: ① $V = \mathcal{P}_n$ has basis $B = \{1, x, x^2, \ldots, x^n\}$
- Spanning is clear
- <u>LI</u>: If $b(x) = b_0 + b_1 x + \cdots + b_n x^n = 0$ zero poly.

Then
$$\begin{aligned} b(0) &= b_0 = 0 \\ b'(0) &= b_1 = 0 \\ b''(0) &= 2b_2 = 0 \\ b'''(0) &= 6b_3 = 0 \\ &\vdots \\ b^{(n)}(0) &= n(n-1)\cdots 2\cdot 1\, b_n = 0 \end{aligned}$$
implies $b_0 = b_1 = b_2 = \cdots = b_n = 0$

② $\{\sin(x), \cos(x)\}$ is a linearly indep set in $\mathcal{F}$

<u>Soln</u>: Write $a \sin x + b \cos x = 0$ (const zero function)

Setting $x = 0$ gives $a \cdot 0 + b \cdot 1 = 0$ so $b = 0$

Setting $x = \frac{\pi}{2}$ ($= 90°$) gives $a \cdot 1 + b \cdot 0 = 0$ so $a = 0$

⚠ $\mathcal{F} = \{$functions of 1 variable$\}$ doesn't have a finite basis

Ditto for $C[0,1]$.

③ Exercise: Find a basis for $W = \{A \in Mat_{3\times3} : A^T = A\}$.

<u>Soln</u>: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ in W means $a_{12}=a_{21}$, $a_{13}=a_{31}$, $a_{23}=a_{32}$

Typical element in W:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{13}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{23}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + a_{33}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(6 values for the entries)

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Then $Sp(S) = W$

Same calculation says S is l.i., so S is a <u>basis</u> for W.